Exam Stochastic Processes 2WB08 - March 10, 2009, 14.00-17.00

The number of points that can be obtained per exercise is mentioned between square brackets. The maximum number of points is 40. Good luck!!

1. (a) [2 pts.] Let $X_1, X_2, \ldots$ be random variables such that the partial sums $S_n = \sum_{i=1}^n X_i$ determine a martingale. Show that $\mathbb{E}(X_i X_j) = 0$ if $i \neq j$.

   Solution: Suppose $i < j$ and remark that $X_j = S_j - S_{j-1}$. Then

$$\begin{aligned} \mathbb{E}(X_i X_j) &= \mathbb{E}(\mathbb{E}(X_i(S_j - S_{j-1})|S_0, S_1, \ldots, S_{j-1})) \\ &= \mathbb{E}(X_i \mathbb{E}((S_j - S_{j-1})|S_0, S_1, \ldots, S_{j-1})) \\ &= \mathbb{E}(X_i(S_{j-1} - S_{j-1})) = 0 \ . \end{aligned}$$

   (b) [3 pts.] Let $(Z_n)_{n\geq 0}$ be the size of the nth generation of an ordinary branching process with $Z_0 = 1$ and with family size Z_1 having mean $\mathbb{E}(Z_1) = \mu$ and variance $\mathrm{Var}(Z_1) = \sigma^2$. Using the properties of conditional expectation, compute the mean $\mathbb{E}(Z_n)$ and show that the variance is given by

$$\mathrm{Var}(Z_n) = \begin{cases} n\sigma^2 & \text{if} \quad \mu = 1 \\ \frac{\sigma^2(\mu^n - 1)\mu^{n-1}}{\mu - 1} & \text{if} \quad \mu \neq 1. \end{cases}$$

   Hint: Recall that $Z_n = \sum_{i=1}^{Z_{n-1}} Y_i$, where $(Y_i)_{i\geq 1}$ is a sequence of i.i.d. random variable with $\mathbb{E}(Y_1) = \mu$ and $\mathrm{Var}(Y_1) = \sigma^2$.

   Solution: We have

$$\begin{aligned} \mathbb{E}(Z_n) &= \mathbb{E}\left(\sum_{i=1}^{Z_{n-1}} Y_i\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{Z_{n-1}} Y_i | Z_{n-1}\right)\right) \\ &= \mathbb{E}(Z_{n-1}\mathbb{E}(Y_1)) \\ &= \mu\mathbb{E}(Z_{n-1}) \ . \end{aligned}$$

   By iteration it is found $\mathbb{E}(Z_n) = \mu^n$.
   A similar computation for the second moment $\mathbb{E}(Z_n^2)$ yields the recursion relation

$$\mathrm{Var}(Z_n) = \mu^2 \mathrm{Var}(Z_{n-1}) + \sigma^2 \mu^{n-1}$$

   which is solved by the claimed expression for $\mathrm{Var}(Z_n)$.

   (c) [3 pts.] Let η be the probability that the branching process ultimately becomes extinct. Define $M_n^{(1)} = \mu^{-n} Z_n$ and $M_n^{(2)} = \eta^{Z_n}$. Check that both the processes $(M_n^{(1)})_{n\geq 1}$ and $(M_n^{(2)})_{n\geq 1}$ are martingales.

   Hint: for $M_n^{(2)}$ use that η is given by the smallest non-negative root of the equation $s = G(s)$, where $G(s) = \mathbb{E}(s^{Z_1})$ is the probability generating function of the family size.

   Solution: This was done during the lecture.

   (d) [2 pts.] Using the martingale $M_n^{(1)}$, show that $\mathbb{E}(Z_n Z_m) = \mu^{n-m}\mathbb{E}(Z_m^2)$ for $m \leq n$. Hence find the correlation coefficient $\rho(Z_m, Z_n)$ in term of μ (you might use the result of item (b)). We recall the definition

$$\rho(Z_m, Z_n) = \frac{\mathrm{cov}(Z_m, Z_n)}{\sqrt{\mathrm{var}(Z_m)\mathrm{var}(Z_n)}}$$

Solution: Using the martingale $M_n^{(1)}$ we immediately have $\mathbb{E}(Z_n|Z_m) = Z_m\mu^{n-m}$. Hence $\mathbb{E}(Z_mZ_n|Z_m) = Z_m^2\mu^{n-m}$ and $\mathbb{E}(Z_mZ_n) = \mathbb{E}(\mathbb{E}(Z_mZ_n|Z_m)) = \mathbb{E}(Z_m^2)\mu^{n-m}$. Thus we have

$$\mathrm{cov}(Z_m, Z_n) = \mathbb{E}(Z_m^2)\mu^{n-m} - \mathbb{E}(Z_n)\mathbb{E}(Z_m) = \mu^{n-m}\mathrm{Var}(Z_m)\ ,$$

and, by the result of item (b),

$$\rho(Z_m, Z_n) = \begin{cases} \sqrt{\mu^{n-m}(1-\mu^m)/(1-\mu^n)} & \text{if} \quad \mu \neq 1 \\ \sqrt{m/n} & \text{if} \quad \mu = 1 \end{cases}$$

2. Let $(B_t)_{t\geq 0}$ be a standard Wiener process and let $(X_t)_{t\geq 0}$ be a standard Brownian motion process with drift $\mu > 0$.

(a) [1 pt.] For $\theta \in \mathbb{R}$ show that the process $(M_t)_{t\geq 0}$ defined by $M_t = \exp\left(\theta B_t - \frac{\theta^2 t}{2}\right)$ is a martingale.

Solution: This has been proved during the lecture.

(b) [3 pts.] Let $a > 0$ and $T_a = \inf\{t \geq 0 : B_t = a\}$. By making use of the martingale in the previous item, compute the Laplace transform $\mathbb{E}(e^{-\lambda T_a})$ where $\lambda > 0$.

Solution: T_a is a stopping time and the martingale stopping theorem can be applied (why?) to the martingale M_t. We have

$$1 = \mathbb{E}\left(\exp\left(\theta B_{T_a} - \frac{\theta^2 T_a}{2}\right)\right)$$

Remark that $B_{T_a} = a$. Thus, choosing $\theta = \sqrt{2\lambda}$, we find

$$\mathbb{E}(e^{-\lambda T_a}) = e^{-\sqrt{2\lambda}\, a}$$

(c) [2 pts.] Conclude from (b) that $\mathbb{E}((T_a)^{-1}) = a^{-2}$.
Hint: Use the identity $x^{-1} = \int_0^\infty e^{-\lambda x} d\lambda$ for $x > 0$.
Solution:

$$\mathbb{E}\left(\frac{1}{T_a}\right) = \int_0^\infty \mathbb{E}(e^{-\lambda T_a}) d\lambda = \int_0^\infty e^{-\sqrt{2\lambda}\, a} d\lambda = \frac{1}{a^2}.$$

(d) [2 pts.] Let $a > 0$ and $\tau_a = \inf\{t \geq 0 : X_t = a\}$. Use again the above martingale to show that the following expression holds

$$\mathbb{E}(e^{-\lambda\tau_a}) = e^{-a(\sqrt{\mu^2+2\lambda}-\mu)}$$

where $\lambda > 0$.

Solution: Notice that $X_t = B_t + \mu t$ and use the stopping theorem as in item (b). The final result is obtained by choosing $\theta = \sqrt{\mu^2 + 2\lambda} - \mu$.

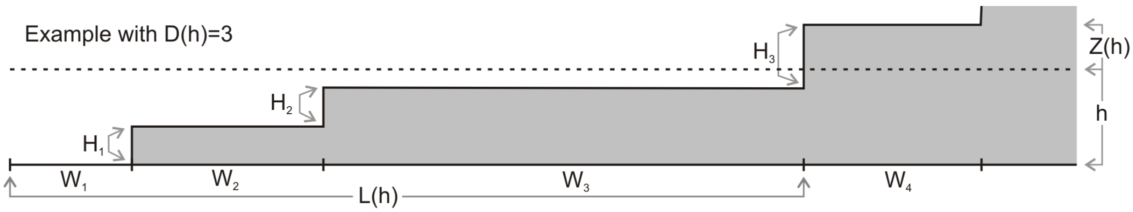
(e) [2 pts.] Compute the mean $\mathbb{E}(\tau_a)$ and the variance $\mathrm{Var}(\tau_a)$.
Solution: Define $g(\lambda) = \mathbb{E}(e^{-\lambda\tau_a})$. Then

$$\mathbb{E}(\tau_a) = -\lim_{\lambda\to 0^+} \frac{dg(\lambda)}{d\lambda}$$

Starting from the $g(\lambda)$ of the previous item, an immediate computation yields $\mathbb{E}(\tau_a) = \frac{x}{\mu}$. A similar computation with the second derivative of $g(\lambda)$ allows to compute the second moment, from which it is found $\mathrm{Var}(\tau_a) = \frac{x}{\mu^3}$.

3. A staircase is constructed such that the kth stair has width W_k and height H_k, where $W_1, H_1, W_2, H_2, \dots$ are nonnegative i.i.d. random variables with distribution function

$$F(x) = \mathbb{P}(H_1 \le x) = \mathbb{P}(W_1 \le x) = 1 - \frac{20}{(1+x)^2}, \quad x \ge 0.$$



(a) [1 pts.] Show that $\mathbb{E}[H_1] = 20$.
Solution: We have

$$\mathbb{E}[H_1] = \int_0^\infty (1 - F(u))\, du = \int_0^\infty \frac{20}{(1+u)^2}\, du = 20 \cdot \int_1^\infty \frac{1}{w^2}\, dw = 20.$$

(b) [3 pts.] Let $D(h)$ be the number of stairs needed to reach at least a total height h. Calculate the limit $\lim_{h\to\infty} \mathbb{E}[D(h)]/h$.
Solution: The successive heights $H_1, H_2, \dots$ form a renewal process with renewal function $\mathbb{E}[D(H)]$, hence

$$\lim_{h\to\infty} \frac{\mathbb{E}[D(h)]}{h} = \frac{1}{\mathbb{E}[H_1]} = \frac{1}{20}.$$

(c) [3 pts.] Let $L(h) = \sum_{k=1}^{D(h)} W_k$ denote the total length of the construction. Derive the limit of the steepness $\mathbb{E}[L(h)]/h$ of the stairway as the height h tends to ∞.
Solution: Consider the renewal process from (a). Taking $W_1, W_2, \dots$ as rewards, we see that $L(h)$ is the total reward. We have from the renewal reward theorem

$$\lim_{t\to\infty} \mathbb{E}[L(h)]/h = \frac{\mathbb{E}[W_1]}{\mathbb{E}[H_1]} = 1.$$

(d) [3 pts.] Find the equilibrium distribution $\tilde{F}_e(x)$ of $F(x)$. Assuming that h is very large, give an estimate for the probability $\mathbb{P}(Z(h) \ge 3)$ for the overshoot $Z(h)$ over the height h (see figure). *Solution:*

$$\tilde{F}_e(x) = \frac{\int_0^x \mathbb{P}(H_1 > u)\, du}{\mathbb{E}[H_1]} = \frac{\int_0^x \frac{20}{(1+u)^2}\, du}{20} = \int_1^{x+1} \frac{1}{u^2}\, du = \frac{x}{1+x}.$$

$Z(h)$ is equivalent to the time to the next event in a renewal process with interarrival times $H_1, H_2, \dots$. We know from the lecture, that the distribution converges to the equilibrium distribution $\tilde{F}_e(x)$. Hence, if h is large, $\mathbb{P}(Z(h) > 20) \approx 1 - \frac{3}{4} = \frac{1}{4}$.

Hint: Note that $H_1, H_2, \dots$ form a renewal process.

4. Buses arrive at a bus stop according to a renewal process with interarrival times $X_1, X_2, \dots$ (measured in minutes), so that the first bus arrives at time X_1. We assume that the $X_1, X_2, \dots$ are i.i.d. with $\mathbb{E}[X_1] < \infty$ and $\mathbb{E}[X_1^2] < \infty$.

(a) [2 pts.] Some busses are green, the others are red. The probability that an arbitrary bus is red is given by $p \in (0,1]$. The color of a bus is independent from the color of the other busses and independent of $X_1, X_2, \ldots$. Let τ be the time of the first arrival of a red bus. Show that $\mathbb{E}[\tau] = \mathbb{E}[X_1]/p$.

Hint: What is the distribution of the number of green busses arriving before τ?

Solution: We have $K \in \{0,1,\ldots\}$ and has a geometric distribution with success probability p and $\mathbb{E}[K] = (1-p)/p$. Since K is independent of $X_1, X_2, \ldots$ it follows from Wald's equation that

$$\mathbb{E}[\tau] = \mathbb{E}[\sum_{k=1}^{K} X_k] + \mathbb{E}[X_1] = \mathbb{E}[K]\mathbb{E}[X_1] + \mathbb{E}[X_1] = \frac{\mathbb{E}[X_1]}{p}.$$

(b) [2 pts.] Let M_t denote the number of red busses minus the number of green busses that arrived during the time interval $(0,t]$. Show that

$$\lim_{t\to\infty} \frac{\mathbb{E}[M_t]}{t} = \frac{2p-1}{\mathbb{E}[X_1]}.$$

Solution: Let R_t denote the number of red busses. Then

$$\lim_{t\to\infty} \frac{\mathbb{E}[R_t]}{t} = \frac{1}{\mathbb{E}[\tau]} = \frac{p}{\mathbb{E}[X_1]}$$

and similarly for G_t, the number of green busses,

$$\lim_{t\to\infty} \frac{\mathbb{E}[G_t]}{t} = \frac{1-p}{\mathbb{E}[X_1]}.$$

Then

$$\lim_{t\to\infty} \frac{\mathbb{E}[R_t - G_t]}{t} = \frac{1}{\mathbb{E}[\tau]} = \frac{2p-1}{\mathbb{E}[X_1]}.$$

(c) [3 pts.] Assume that people arrive with constant rate 1 $person/min$ to the bus stop. At the time a bus arrives, all waiting passengers receives $1/p$ Euros if the bus is red and $1/(1-p)$ Euros if the bus is green. Let C_t denote the total amount of money spend after time t by the bus company. Show that

$$\lim_{t\to\infty} \frac{\mathbb{E}[C_t]}{t} = 2.$$

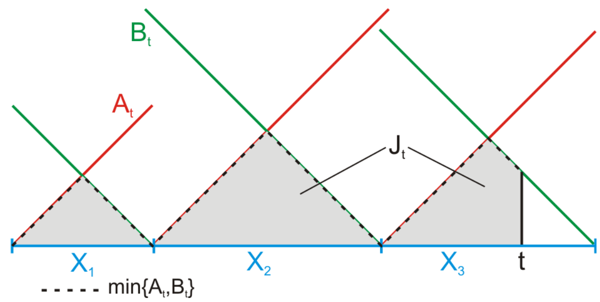
Solution: Let C_k denote the money that the company has to pay when bus k arrives. Then $\mathbb{E}[C_k] = (\frac{1}{p} \cdot p + \frac{1}{1-p}(1-p))\mathbb{E}[X_1] = 2\mathbb{E}[X_1]$ and by the renewal reward theorem

$$\lim_{t\to\infty} \frac{\mathbb{E}[C_t]}{t} = \frac{\mathbb{E}[C_k]}{\mathbb{E}[X_1]} = 2.$$

(d) [3 pts.] At time t let A_t denote the time that went by since the last bus arrived and B_t the time until the next bus arrives. Sketch the process $J_t = \int_0^t \min\{A_s, B_s\}\, ds$. By using an appropriate renewal reward process, show that

$$\lim_{t\to\infty} \frac{\mathbb{E}[J_t]}{t} = \frac{\mathbb{E}[X_1^2]}{4\mathbb{E}[X_1]}.$$

Solution:



Let J_{T_k} be the reward at time T_k, where $T_k = X_1 + X_2 + \ldots + X_k$. Then (draw the process!)

$$\mathbb{E}[J_{T_k}] = \mathbb{E}[\frac{1}{4}X_k^2],$$

since the area under the process $\min\{A_s, B_s\}$ between T_{k-1} and T_k (one cycle) is equal to one quarter of the square with area $X_k \times X_k$. The result follows from renewal theory for reward processes.