Exam Stochastic Processes 2WB08 - January 15, 2009, 14.00-17.00

The number of points that can be obtained per exercise is mentioned between square brackets. The maximum number of points is 40. Good luck!!

1. Let X and Z be two random variables. We define $Var(X|Z)$ as

$$Var(X|Z) = \mathbb{E}(X^2|Z) - (\mathbb{E}(X|Z))^2$$

   (a) [3 pts.] Show that the following identity holds

$$Var(X) = \mathbb{E}(Var(X|Z)) + Var(\mathbb{E}(X|Z))$$

   Solution: We have

$$\begin{aligned}\mathbb{E}(Var(X|Z)) &= \mathbb{E}(\mathbb{E}(X^2|Z)) - \mathbb{E}((\mathbb{E}(X|Z))^2) \\ &= \mathbb{E}(X^2) - \mathbb{E}((\mathbb{E}(X|Z))^2)\end{aligned}$$

   and

$$\begin{aligned}Var(\mathbb{E}(X|Z)) &= \mathbb{E}((\mathbb{E}(X|Z))^2) - (\mathbb{E}(\mathbb{E}(X|Z)))^2 \\ &= \mathbb{E}((\mathbb{E}(X|Z))^2) - (E(X))^2\end{aligned}$$

   Adding up (side by side) the previous equations the claim is proved.

   (b) [3 pts.] Using the previous result prove the following: if $Y_1, Y_2, \ldots$ are i.i.d. random variables and N is an independent integer valued random variable and $X = Y_1 + Y_2 + \ldots + Y_N$ then

$$Var(X) = E(N)Var(Y_1) + (E(Y_1))^2 Var(N)$$

   Solution: We apply the previous result with $X = Y_1 + Y_2 + \ldots + Y_N$ and $Z = N$. We have

$$\begin{aligned}\mathbb{E}(Var(X|N)) &= \mathbb{E}(\mathbb{E}(X^2|N)) - \mathbb{E}((\mathbb{E}(X|N))^2) \\ &= \mathbb{E}(N\mathbb{E}(Y_1^2) + N(N-1)\mathbb{E}(Y_1Y_2)) - \mathbb{E}(N^2(\mathbb{E}(Y_1))^2) \\ &= \mathbb{E}(N)Var(Y_1)\end{aligned}$$

   and

$$\begin{aligned}Var(\mathbb{E}(X|N)) &= \mathbb{E}((\mathbb{E}(X|N))^2) - (\mathbb{E}(\mathbb{E}(X|N)))^2 \\ &= \mathbb{E}(N^2(\mathbb{E}(Y_1))^2) - (\mathbb{E}(N\mathbb{E}(Y_1)))^2 \\ &= \mathbb{E}(N^2)(\mathbb{E}(Y_1))^2 - (\mathbb{E}(N))^2(\mathbb{E}(Y_1))^2 \\ &= (\mathbb{E}(Y_1))^2 Var(N)\end{aligned}$$

   Adding up (side by side) the previous equations the claim is proved.

(c) [2 pts.] Consider the "random harmonic serie" $M_n = \sum_{j=1}^{n} \frac{1}{j} X_j$ where $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. Show that $(M_n)_{n\in\mathbb{N}}$ (with $M_0 = 0$) is a discrete time martingale process.

Solution: It is enough to show that M_n is a martingale with respect to the sequence of random variables $(X_i)_{i\in\mathbb{N}}$. We have $E(|M_n|) < \infty$ and

$$\mathbb{E}(M_{n+1}|X_1, \ldots, X_n) = M_n + \frac{1}{n+1}\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) = M_n$$

because the X_i's are centered i.i.d. random variables.

(d) [2 pts.] It is known that the harmonic serie $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent, while the alternating harmonic serie $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is convergent. What would you say on the convergence property of the "random harmonic serie"? Study the convergence in the almost sure sense.

Solution: The martingale is L_2-bounded. Indeed

$$\mathbb{E}(M_n^2) = \mathbb{E}(\sum_{k=1}^{n}\sum_{j=1}^{n} \frac{1}{k} X_k \frac{1}{j} X_j) = \sum_{j=1}^{n} \frac{1}{j^2} < \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

By martingale convergence theorem the process $(M_n)_{n\in\mathbb{N}}$ has a limit and the limit is finite with probability one. Thus the "random harmonic serie" is convergent in the almost sure sense.

2. Let $(B_t)_{t\geq 0}$ be a standard Wiener process.

(a) [1 pt.] For $\theta \in \mathbb{R}$ show that the process $(M_t)_{t\geq 0}$ defined by $M_t = \exp\left(\theta B_t - \frac{\theta^2 t}{2}\right)$ is a martingale.

Solution: This has been proved during the lecture.

(b) [2 pts.] Let $a > 0$ and $T = \inf\{t \geq 0 : B_t \notin (-a, a)\}$. By making use of the martingale in the previous item, compute the Laplace transform $\mathbb{E}(e^{-\lambda T})$ where $\lambda > 0$.

Solution: T is a stopping time and the martingale stopping theorem can be applied (why?) to the martingale M_t. We have

$$1 = \mathbb{E}\left(\exp\left(\theta B_T - \frac{\theta^2 T}{2}\right)\right)$$

By symmetry $\mathbb{P}(B_T = a) = \mathbb{P}(B_T = -a) = 1/2$ and B_T is independent of T. Thus

$$1 = \cosh(\theta a)\mathbb{E}\left(\exp\left(-\frac{\theta^2 T}{2}\right)\right)$$

Take $\theta = \sqrt{2\lambda}$ to find

$$\mathbb{E}(e^{-\lambda T}) = \frac{1}{\cosh(a\sqrt{2\lambda})}$$

(c) [2 pts.] Use the result of the previous item to compute the expected time $\mathbb{E}(T)$.
Solution: Define $g(\lambda) = \mathbb{E}(e^{-\lambda T})$. Then

$$\mathbb{E}(T) = -\lim_{\lambda \to 0^+} \frac{dg(\lambda)}{d\lambda}$$

Starting from the $g(\lambda)$ of the previous item, an immediate computation yields $\mathbb{E}(T) = a^2$. The same result was obtained during the lecture by applying the martingale stopping theorem to the martingale $B_t^2 - t$.

(d) [3 pts.] Compute the probability $\mathbb{P}(B_2 > 0 | B_1 > 0)$. *Hint:* to compute an integral use polar coordinates.
Solution: From the definition of conditional probability it follows

$$\mathbb{P}(B_2 > 0 | B_1 > 0) = \frac{\mathbb{P}(B_2 > 0, B_1 > 0)}{\mathbb{P}(B_1 > 0)}$$

The denominator equal 1/2 by symmetry. The numerator is given by

$$\mathbb{P}(B_2 > 0, B_1 > 0) = \frac{1}{2\pi} \int_0^{+\infty} dx_1 \int_0^{+\infty} dx_2 \ \exp\left(-\frac{x_1^2}{2}\right) \exp\left(-\frac{(x_2 - x_1)^2}{2}\right)$$

To compute this integral we first make the change of variables $x = x_1$, $y = x_2 - x_1$

$$\mathbb{P}(B_2 > 0, B_1 > 0) = \frac{1}{2\pi} \int_0^{+\infty} dx \int_{-x}^{+\infty} dy \ \exp\left(-\frac{x^2}{2}\right) \exp\left(-\frac{y^2}{2}\right)$$

and then we go to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$\mathbb{P}(B_2 > 0, B_1 > 0) = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/2} d\theta \int_0^{+\infty} dr \ r \exp\left(-\frac{r^2}{2}\right) = \frac{3}{8}$$

Thus we have

$$\mathbb{P}(B_2 > 0 | B_1 > 0) = \frac{\mathbb{P}(B_2 > 0, B_1 > 0)}{\mathbb{P}(B_1 > 0)} = \frac{3}{4}$$

(e) [2 pts.] Are the events $B_1 > 0$ and $B_2 > 0$ independent?
Solution: No, because $\mathbb{P}(B_2 > 0 | B_1 > 0) \neq \mathbb{P}(B_2 > 0)$.

3. Consider a renewal process $N(t)$ with X_k denoting the time between the $(k-1)$th and the kth event and suppose that $\mathbb{E}(X_1) < \infty$.

(a) [2 pts.] Let $K(t,s)$ be equal to the number of events during the time interval $(t, t+s]$ divided by the duration s. Show, by using the results from the lecture, that, if $F(x) = \mathbb{P}(X_1 \leq x)$ is nonlattice,

$$\lim_{t \to \infty} \mathbb{E}(K(t,s)) = \lim_{s \to \infty} \mathbb{E}(K(t,s)) = \frac{1}{\mathbb{E}(X_1)}.$$

Solution: We have $K(t,s) = (N(t+s) - N(t))/s$ and hence

$$\lim_{t \to \infty} \mathbb{E}(K(t,s)) = \frac{m(t+s) - m(t)}{s} = \frac{1}{\mathbb{E}(X_1)}$$

by Blackwell's theorem. Moreover

$$\lim_{s\to\infty} \mathbb{E}(K(t,s)) = \frac{m(t+s)}{t+s}\frac{t+s}{s} - \frac{m(t)}{s} = \frac{1}{\mathbb{E}(X_1)}$$

by the elementary renewal theorem.

(b) [3 pts.] Let $M(t) = \frac{1}{N(t)}\sum_{k=1}^{N(t)}(X_k - \mathbb{E}(X_k))^2$ and assume that $\mathbb{E}(X_1^2) < \infty$. Show that $\lim_{t\to\infty} M(t) = Var(X_1)$ with probability one.

<u>*Solution:*</u> We know that $t/N(t) \to \mathbb{E}(X_1)$. Define a renewal reward process with $R_k = (X_k - \mathbb{E}(X_k))^2$, then by the renewal reward theorem

$$\frac{R(t)}{t} \to \frac{\mathbb{E}((X_k - \mathbb{E}(X_k))^2)}{\mathbb{E}(X_1)} = \frac{Var(X_2)}{\mathbb{E}(X_1)}.$$

Note that $(X_1 - \mathbb{E}(X_1))^2 < \infty$ follows from $\mathbb{E}(X_1^2) < \infty$. Consequently

$$\lim_{t\to\infty} M(t) \quad = \quad \lim_{t\to\infty} \frac{R(t)}{N(t)} = \lim_{t\to\infty} \frac{R(t)}{t} \cdot \frac{t}{N(t)} = Var(X_1).$$

(c) [3 pts.] Suppose that X_1 is integer valued, i.e. $\mathbb{P}(X_1 = k) = p_k$, with probabilities $p_k \in [0,1]$ and $p_1 + p_2 + \ldots = 1$. Show, by conditioning on X_1, that the renewal function satisfies the discrete renewal equation

$$m(n) = \sum_{k=0}^{n}(1 + m(n-k)) \cdot p_k \quad , \quad n = 1, 2, \ldots.$$

<u>*Solution:*</u> We have

$$\begin{aligned} m(n) &= \mathbb{E}(N(n)) = \sum_{k=0}^{\infty} \mathbb{E}(N(n)|X_1 = k)\mathbb{P}(X_1 = k) \\ &= \sum_{k=0}^{n} \mathbb{E}(1 + N(n-k))p_k. \end{aligned}$$

(d) [2 pts.] Suppose that $\mathbb{P}(X_1 = 0) = 1 - p$ and $\mathbb{P}(X_1 = 1) = p$ for some $p \in (0,1]$. Find an explicit formula for $m(n)$.

<u>*Solution:*</u> We have

$$m(0) = (1 + m(0))(1-p)$$

so $m(0) = \frac{1-p}{p}$. Moreover

$$m(n) = (1 + m(n))(1-p) + p(1 + m(n-1)),$$

so that

$$m(n) = \frac{1}{p} + m(n-1) = \frac{2}{p} + m(n-2) = \ldots = \frac{n+1-p}{p}.$$

4. A $G/M/1/1$ queueing system consists of a single server with capacity one. Arriving (potential) customers are rejected and leave immediately if there is already a customer in service. If an arriving customer finds the system empty, he/she enters immediately and leaves the system as soon as the service is completed. We assume that the service times are exponential with mean $1/\lambda$ and the interarrival times are given by i.i.d. random variables $Z_1, Z_2, \ldots$, with distribution function $F(x)$, mean $\mu < \infty$ and Laplace transform $\phi(\lambda) = \mathbb{E}(e^{-\lambda Z_1}) = \int_0^\infty e^{-\lambda z}\, dF(z)$. We suppose that the first customer arrives at time 0, so that Z_1 denotes the time between the first and the second arrival.

   (a) [3 pts.] Let T be the time, when the second service begins and let C be the number of customers that arrive during $(0, T]$. Show that

$$\mathbb{P}(C = k) = \phi(\lambda)^{k-1}(1 - \phi(\lambda)) \quad , \quad k = 1, 2, \ldots$$

   Solution: The probability p that the second customer arrives after the completion of the first service is given by (we condition on the length of the first service time)

$$\int_0^\infty (1 - F(z))\lambda e^{-\lambda z}\, dz = 1 - \int_0^\infty e^{-\lambda z}\, dF(z) = 1 - \phi(\lambda).$$

   If the second customer arrives before the first service is completed then the remaining (first) service is again exponentially distributed. It follows that the number M of arriving customers during the first service time is geometric,

$$\mathbb{P}(M = k) = (1 - p)^k p \quad , \quad k = 0, 1, \ldots,$$

   and the result follows from the fact that $C = M + 1$.

   (b) [3 pts.] Show that the Laplace transform and the mean of T are given by

$$\mathbb{E}(e^{-sT}) = \frac{(1 - \phi(\lambda))\phi(s)}{1 - \phi(\lambda)\phi(s)} \quad \text{and} \quad \mathbb{E}(T) = \frac{\mu}{1 - \phi(\lambda)}.$$

   Hint: express T in terms of $Z_1, Z_2, \ldots$ and C.
   Solution: Since $T = Z_1 + \ldots + Z_C$, we have

$$\begin{aligned}
\mathbb{E}(e^{-sT}) &= \sum_{k=1}^\infty \mathbb{E}(e^{-sT} | C = k)\mathbb{P}(C = k) \\
&= (1 - \phi(\lambda)) \sum_{k=1}^\infty \mathbb{E}(e^{-s(Z_1 + Z_2 + \ldots + Z_k)})\phi(\lambda)^{k-1} \\
&= \frac{1 - \phi(\lambda)}{\phi(\lambda)} \sum_{k=1}^\infty \phi(s)^k \phi(\lambda)^k = \frac{(1 - \phi(\lambda))\phi(s)}{1 - \phi(s)\phi(\lambda)}.
\end{aligned}$$

   It follows that

$$\begin{aligned}
\mathbb{E}(T) &= -\frac{d}{ds}\frac{(1 - \phi(\lambda))\phi(s)}{1 - \phi(s)\phi(\lambda)}\Big|_{s=0.} \\
&= -(1 - \phi(\lambda))\frac{\phi'(s)(1 - \phi(s)\phi(\lambda)) + \phi'(s)\phi(s)\phi(\lambda)}{(1 - \phi(s)\phi(\lambda))^2}\Big|_{s=0.} = \frac{\mu}{1 - \phi(\lambda)}.
\end{aligned}$$

(c) [4 pts.] Let $W(t)$ denote the number of customers in the system. Find the limit distribution $\lim_{t\to\infty} \mathbb{P}(W(t) = 1)$.

Solution: $W(t)$ defines an alternating renewal process, switching from on $(W(t) = 1)$ to off $(W(t) = 0)$ whenever a customer leaves the system. According to the limit theorem for alternating renewal processes we have

$$\lim_{t\to\infty} \mathbb{P}(W(t) = 1) = \frac{\mathbb{E}(X)}{\mathbb{E}(X+Y)} = \frac{1-\phi(\lambda)}{\lambda\mu}.$$