## Exam Stochastic Processes 2WB08 - March 13, 2007, 14.00-17.00

The number of points that can be obtained per exercise is mentioned between square brackets. The maximum number of points is 40. Good luck!!

**Problem 1:** Consider a renewal process with distribution $F(\cdot)$ of the times between successive renewals. Let $m(t)$ denote the renewal function of this process.
a) [3 pt.] Argue that $m(t)$ satisfies the following equation:

$$m(t) = F(t) + \int_0^t m(t-x)\mathrm{d}F(x), \quad t \geq 0.$$

Let $F(t) = 1 - \mathrm{e}^{-t/\mu}$, $t \geq 0$.
b) [2 pt.] Derive an expression for $m(t)$.
c) [2 pt.] Derive an expression for the expectation of the time of occurrence of the first renewal after $t$.
d) [3 pt.] Use alternating renewal processes to derive an expression for the limiting distribution of the residual time $Y(t)$ from $t$ until the first renewal after $t$: $\lim_{t\to\infty} P(Y(t) \leq x)$.

**Problem 2:** Let $(S_n)_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$, i.e. $S_n = \sum_{i=1}^n X_i$ ($S_0 = 0$), with $(X_i)_{i\geq 0}$ a sequence of i.i.d. random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. For a fixed $a \in \mathbb{N}$ define

$$T = \min\{n \in \mathbb{N} : |S_n| = a\}.$$

a) [2 pt.] Show that $M_n = S_n^2 - n$ and $N_n = S_n^4 - 6nS_n^3 + 3n^2 + 2n$ are martingales.
b) [2 pt.] State the martingale stopping theorem.
c) [2 pt.] Show that $\mathbb{E}(T) = a^2$ and $\mathbb{E}(T^2) = \frac{a^2}{3}(5a^2 - 2)$.

Consider now the case of a biased random walk, namely $\mathbb{P}(X_i = 1) = p > \frac{1}{2}$ and $\mathbb{P}(X_i = -1) = q = 1 - p < \frac{1}{2}$. Define $Y_n = e^{bS_n - cn}$ for constants $b$ and $c$. Define also

$$T_1 = \min\{n \in \mathbb{N} : S_n = 1\}.$$

d) [2 pt.] Derive a necessary relation between the constants $b$ and $c$ in order that $Y_n$ is a martingale.
e) [2 pt.] Find the moment generating function $\mathbb{E}(e^{-cT_1})$ for $c > 0$.

**Problem 3:** Consider the simple random walk: $P(X_i = 1) = p$, $P(X_i = -1) = 1 - p$, $i = 1, 2, \ldots$, and $S_n = \sum_{i=1}^n X_i$, $n = 1, 2, \ldots$, with $S_0 = 0$. $X$ has the same distribution as the $X_i$. Suppose $0 < p < \frac{1}{2}$. Let $\theta \neq 0$ be such that $E[\mathrm{e}^{\theta X}] = 1$.
a) [2 pt.] Give the distribution of $S_n$.
b) [2 pt.] Argue that $\{Z_n,\ n = 0, 1, \ldots\}$ with $Z_n = \mathrm{e}^{\theta S_n}$ is a martingale.
c) [2 pt.] Let $A > 0, B > 0$ and define the stopping time $N$ as $N = \min\{n : S_n = A \textit{ or } S_n = -B\}$. Find an expression for the probability $P_A$ that the random walk reaches $A$ before it reaches $-B$.
d) [2 pt.] Use Jensen's inequality ($E[f(X)] \geq f(E[X])$ for $f$ convex) to show that $\theta > 0$.
e) [2 pt.] Prove that the probability that the random walk ever reaches $A$ is bounded by $\mathrm{e}^{-\theta A}$.

**Problem 4:** Let $\{X(t), t \geq 0\}$ be a standard Brownian motion and let $M(t) = \max_{0 \leq s \leq t} X(s)$. Define

$$Y(t) = \exp\{cX(t) - c^2 t/2\}, \quad t \geq 0$$

where $c$ is any constant.
a) [2 pt.] Define $X(t)$ is a standard Brownian motion.
b) [2 pt.] Show that $\{Y(t), t \geq 0\}$ is a martingale with mean 1.
c) [2 pt.] Is $Y(t)$ a Brownian motion? Motivate your answer.
d) [2 pt.] Derive the distribution of $M(t) - X(t)$.
e) [2 pt.] Show that

$$\mathbb{P}(M(t) > a | M(t) = X(t)) = \exp -a^2/2t, \qquad a > 0.$$

**Solution problem 2:**
a) The integrability condition is trivial for both $M_n$ and $N_n$. Since $S_{n+1} = S_n + X_{n+1}$, an immediate computation shows that

$$\mathbb{E}(S_{n+1}^2|X_0,\ldots,X_n) = S_n^2 + 1$$

and

$$\mathbb{E}(S_{n+1}^4|X_0,\ldots,X_n) = S_n^4 + 6S_n^2 + 1$$

From these it follows the martingale property easily, both for $M_n$ and $N_n$.
b) See Ross book, Th. 6.2.2, page 300.
c) The martingale stopping theorem is applicable to both martingales $M_n$ and $N_n$ with the stopping time $T$. Indeed we have $|S_{T\wedge n}| \le a \ \forall n \in \mathbb{N}$ and one can also prove that $\mathbb{P}(T < \infty) = 1$ (this was done during the lecture). We then have $\mathbb{E}(M_T) = \mathbb{E}(M_0) = 0$ and $\mathbb{E}(N_T) = \mathbb{E}(N_0) = 0$. On the other hand

$$\mathbb{E}(M_T) = \mathbb{E}(S_T^2) - \mathbb{E}(T) = a^2 - \mathbb{E}(T)$$

and

$$\mathbb{E}(N_T) = \mathbb{E}(S_T^4) - 6\mathbb{E}(TS_T^2) + 3\mathbb{E}(T^2) + 2\mathbb{E}(T) = a^4 - 6a^2\mathbb{E}(T) + 3\mathbb{E}(T^2) + 2\mathbb{E}(T)\,.$$

From these you derive the claimed results.
d) The integrability condition does not imply any constraint. Indeed, since $|S_n| \le n$

$$\mathbb{E}(|Y_n|) = \mathbb{E}(e^{bS_n - cn}) \le e^{(b-c)n} < \infty \qquad \forall n \in \mathbb{N}$$

For the martingale property to hold we require $\mathbb{E}(Y_{n+1}|X_0,\ldots,X_n) = Y_n$. This implies

$$b = \ln\left(\frac{e^c \pm \sqrt{e^{2c} - 4pq}}{2p}\right)$$

e) If the martingale stopping theorem is applicable to the martingale $Y_n$ with the stopping time $T_1$, it follows that $\mathbb{E}(Y_{T_1}) = \mathbb{E}(Y_0) = 1$. This implies that

$$\mathbb{E}(e^{-cT_1}) = e^{-b}$$

To be the stopped process uniformly bounded we require $b > 0$. In order for this to be true we have to choose the positive root of $pe^{2b} - e^{b+c} + q = 0$. Thus

$$\mathbb{E}(e^{-cT_1}) = e^{-b} = \frac{2p}{e^c + \sqrt{e^{2c} - 4pq}}\,.$$

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