Exam Stochastic Processes 2WB08 - March 23, 2007, 14.00-17.00

The number of points that can be obtained per exercise is mentioned between square brackets. The maximum number of points is 40. Good luck!!

Problem 1: Consider a renewal process with distribution $F(\cdot)$ of the times between successive renewals. Let $m(t)$ denote the renewal function of this process.
a) [3 pt.] Argue that $m(t)$ satisfies the following equation:

$$m(t) = F(t) + \int_0^t m(t-x)\mathrm{d}F(x), \quad t \geq 0.$$

Let $F(t) = 1 - \mathrm{e}^{-t/\mu}$, $t \geq 0$.
b) [2 pt.] Derive an expression for $m(t)$.
c) [2 pt.] Derive an expression for the expectation of the time of occurrence of the first renewal after t.
d) [3 pt.] Use alternating renewal processes to derive an expression for the limiting distribution of the residual time $Y(t)$ from t until the first renewal after t: $\lim_{t\to\infty} P(Y(t) \leq x)$.

Problem 2: Let $(S_n)_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$, i.e. $S_n = \sum_{i=1}^n X_i$ ($S_0 = 0$), with $(X_i)_{i\geq 0}$ a sequence of i.i.d. random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. For a fixed $a \in \mathbb{N}$ define

$$T = \min\{n \in \mathbb{N} : |S_n| = a\}.$$

a) [3 pt.] Show that $M_n = S_n^2 - n$ and $N_n = S_n^4 - 6nS_n^2 + 3n^2 + 2n$ are martingales.
b) [1 pt.] State the martingale stopping theorem.
c) [2 pt.] Show that $\mathbb{E}(T) = a^2$ and $\mathbb{E}(T^2) = \frac{a^2}{3}(5a^2 - 2)$.

Consider now the case of a biased random walk, namely $\mathbb{P}(X_i = 1) = p > \frac{1}{2}$ and $\mathbb{P}(X_i = -1) = q = 1 - p < \frac{1}{2}$. Define $Y_n = e^{bS_n - cn}$ for constants b and c. Define also

$$T_1 = \min\{n \in \mathbb{N} : S_n = 1\}.$$

d) [2 pt.] Derive a necessary relation between the constants b and c in order that Y_n is a martingale.
e) [2 pt.] Find the moment generating function $\mathbb{E}(e^{-cT_1})$ for $c > 0$.

Problem 3: Consider the simple random walk: $P(X_i = 1) = p$, $P(X_i = -1) = 1 - p$, $i = 1, 2, \ldots$, and $S_n = \sum_{i=1}^n X_i$, $n = 1, 2, \ldots$, with $S_0 = 0$. Suppose $0 < p < \frac{1}{2}$ and let $\theta \neq 0$ be such that $E[\mathrm{e}^{\theta X}] = 1$.
a) [2 pt.] Give the distribution of S_n.
b) [2 pt.] Argue that $\{Z_n,\ n = 0, 1, \ldots\}$ with $Z_n = \mathrm{e}^{\theta S_n}$ is a martingale.
c) [2 pt.] Let $A > 0, B > 0$ and define the stopping time N as $N = \min\{n : S_n = A \text{ or } S_n = -B\}$. Find an expression for the probability P_A that the random walk reaches A before it reaches $-B$.
d) [2 pt.] Use Jensen's inequality ($E[f(X)] \geq f(E[X])$ for f convex) to show that $\theta > 0$.
e) [2 pt.] Prove that the probability that the random walk ever reaches A is bounded by $\mathrm{e}^{-\theta A}$.

Problem 4: Let $\{X(t), t \geq 0\}$ be a standard Brownian motion and let $M(t) = \max_{0 \leq s \leq t} X(s)$. Define

$$Y(t) = \exp\{cX(t) - c^2 t/2\}, \quad t \geq 0$$

where c is any constant.
a) [1 pt.] Define $X(t)$ is a standard Brownian motion.
b) [3 pt.] Show that $\{Y(t), t \geq 0\}$ is a martingale with mean 1.
c) [2 pt.] Is $Y(t)$ a Brownian motion? Motivate your answer.
d) [2 pt.] Derive the distribution of $M(t) - X(t)$.
e) [2 pt.] Show that

$$\mathbb{P}(M(t) > a | M(t) = X(t)) = \exp\{-a^2/2t\}, \qquad a > 0.$$