Exam Stochastic Processes 2WB08 - March 25, 2008, 14.00-17.00

Problem 2: Let $(X_i)_{i\geq 1}$ be a i.i.d. sequence of random variables with distribution

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } r \\ -1 & \text{with probability } q \end{cases}$$

where $p, q, r > 0$. Let $(S_n)_{n\geq 1} = S_0 + \sum_{i=1}^n X_i$ be a discrete time random walk starting at $S_0 = m \in \mathbb{N}$ at time zero.
a) [2 pt.] Define $(Y_n)_{n\geq 1} = \left(\frac{q}{p}\right)^{S_n}$. Show that $(Y_n)_{n\geq 1}$ is a martingale and that for any positive integer n one has $\mathbb{E}(Y_n) = \left(\frac{q}{p}\right)^m$.
b) [2 pt.] Let T be the time until the walker reaches either 0 or N for the first time, where N is an integer greater than m. Compute $\mathbb{E}(Y_T)$. If you apply the martingale stopping theorem remember to check that the hypothesis for the applicability of the theorem are satisfied.
c) [3 pt.] Assuming $p \neq q$, compute the probability that, starting from m, the walker reaches 0 before it reaches N.
d) [3 pt.] In the case $p = q$ you may assume that probability of the previous item is $\mathbb{P}(S_T = 0 \mid S_0 = m) = (N - m)/N$. Now define $Z_n = S_n^2 - 2np$. Prove that $(Z_n)_{n\geq 1}$ is a martingale and show that the expected time until absorption is given by

$$\mathbb{E}(T|S_0 = m) = \frac{m(N - m)}{2p}$$

Problem 4: Let $B(t)$ be a standard Wiener process. For $\beta > 0$ and $\sigma > 0$, consider the process

$$Y(t) = e^{-\beta t} \frac{\sigma}{\sqrt{2\beta}} B(e^{2\beta t} - 1) .$$

a) [1 pt.] Show that the distribution of $Y(t)$ is normal $N\left(0, \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})\right)$.
b) [2 pt.] Is $Y(t)$ a Gaussian process? Compute its covariance $Cov(Y(s)Y(t))$.
c) [2 pt.] Show that in the limit $t, s \to \infty$ with finite $|t - s|$ the process becomes stationary. We recall that a process is said to be stationary if $Y(t_1), \ldots, Y(t_n)$ has the same joint distributions of $Y(t_1 + h), \ldots, Y(t_n + h)$ for all $t_1, \ldots, t_n, h, n$.
d) [3 pt.] Compute the instantaneous mean $a(t, y)$ and the instantaneous variance $b(t, y)$ of $Y(t)$ defined as

$$\mathbb{E}(Y(t + h) - Y(t) \mid Y(t) = y) = a(t, y)\, h + o(h) ,$$
$$\mathbb{E}((Y(t + h) - Y(t))^2 \mid Y(t) = y) = b(t, y)\, h + o(h) .$$

e) [2 pt.] Find a *stationary* solution for the Kolmogorov forward differential equation

$$\frac{\partial}{\partial t} p(y, t) = -\frac{\partial}{\partial y}(a(t, y)p(y, t)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(b(t, y)p(y, t))$$

Hint.: This can be obtained by imposing $\frac{\partial}{\partial t} p(y, t) = 0$.

Solution problem 2:
a) The integrability condition $\mathbb{E}(|S_n|) < \infty$ is obvious. It is enough to check that S_n has the martingale property w.r.t. X_n. We have

$$\begin{aligned}
\mathbb{E}\left(Y_{n+1}|X_1,\ldots,X_n\right) &= \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_n+X_{n+1}}|X_1,\ldots,X_n\right) && (1)\\
&= \left(\frac{q}{p}\right)^{S_n}\mathbb{E}\left(\left(\frac{q}{p}\right)^{X_{n+1}}|X_1,\ldots,X_n\right) && (2)\\
&= \left(\frac{q}{p}\right)^{S_n}\left(\frac{q}{p}p+\frac{p}{q}q+r\right) && (3)\\
&= Y_n && (4)
\end{aligned}$$

The expectation of a martingale does not depend on the time, that is $\mathbb{E}(Y_{n+1}) = \mathbb{E}(Y_n)$, as it is immediately seen by taking expectations in the previous relation. This implies that $\mathbb{E}(Y_n) = \mathbb{E}(Y_0) = \left(\frac{q}{p}\right)^m$.

b) T is a stopping time w.r.t. the martingale Y_n and the stopped process is bounded:

$$Y_{n\wedge T} = \left(\frac{q}{p}\right)^{S_{n\wedge T}} = \begin{cases} \left(\frac{q}{p}\right)^N & \text{if } p<q \\ 1 & \text{if } p>q \end{cases}$$

Applying the optional stopping theorem one has

$$\mathbb{E}(Y_T) = \mathbb{E}(Y_0) = (q/p)^m$$

c) Combining together

$$\mathbb{E}(Y_T) = (q/p)^N\mathbb{P}(S_T = N|S_0 = m) + (q/p)^0\mathbb{P}(S_T = 0|S_0 = m)$$

and

$$\mathbb{P}(S_T = N|S_0 = m) + \mathbb{P}(S_T = 0|S_0 = m) = 1$$

one finds that

$$\mathbb{P}(S_T = N|S_0 = m) = \frac{(q/p)^m - (q/p)^N}{1-(q/p)^N}$$

d) We have

$$\begin{aligned}
\mathbb{E}\left(Z_{n+1}|X_1,\ldots,X_n\right) &= \mathbb{E}\left(S_n^2 + 2S_nX_{n+1} + X_{n+1}^2 - 2(n+1)p|X_1,\ldots,X_n\right) && (5)\\
&= S_n^2 + 2S_n\mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) - 2(n+1)p && (6)\\
&= S_n^2 + 2p - 2(n+1)p && (7)\\
&= Z_n && (8)
\end{aligned}$$

Applying the optimal stopping theorem we get

$$\mathbb{E}(Z_T) = \mathbb{E}(Z_0) = m^2$$

and

$$\mathbb{E}(Z_T) = \mathbb{E}(S_T^2) - 2p\mathbb{E}(T) = N^2\mathbb{P}(S_t = N|S_0 = m) - 2p\mathbb{E}(T) = N^2\frac{m}{N} - 2p\mathbb{E}(T)\ .$$

This implies the required expression for $\mathbb{E}(T)$.

Solution problem 4:

a) For a standard Brownian motion one has that $B(t)$ is $N(0,t)$ distributed. For a continuous mapping $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, $B(\phi(t))$ is $N(0,\phi(t))$ distributed. Considering $\phi(t) = e^{2\beta t} - 1$ and observing that $Y(t)$ is a linear transformation of $B(\phi(t))$ the claim immediately follows.

b) Since $B(t)$ is Gaussian it follows that $Y(t)$ is also Gaussian. We know that the covariance of Brownian montion: $\mathbb{E}(B(t)B(s)) = t \wedge s$. The mapping $\phi(t)$ is monotonically increasing. This implies

$$\mathbb{E}(B(\phi(t))B(\phi(s))) = \phi(t) \wedge \phi(s) = e^{2\beta(t\wedge s)} - 1 \,.$$

We then have

$$\begin{aligned}
\mathbb{E}(Y(t)Y(s)) &= \mathbb{E}\left(e^{-\beta t}\frac{\sigma}{\sqrt{2\beta}}\,B(\phi(t))e^{-\beta s}\frac{\sigma}{\sqrt{2\beta}}\,B(\phi(s))\right) \\
&= e^{-\beta(t+s)}\frac{\sigma^2}{2\beta}\,(e^{2\beta(t\wedge s)} - 1) \\
&= \frac{\sigma^2}{2\beta}\left[e^{-\beta|t-s|} - e^{-\beta(t+s)}\right]
\end{aligned}$$

c) Since the process $Y(t)$ is Markov and Gaussian, all finite dimensional joint distributions are obtained from the mean and covariance. Then a necessary and sufficient condition for $Y(t)$ to be stationary is that the mean $\mathbb{E}(Y(t))$ does not depend on t and the $Cov(Y(t), Y(s))$ depends only on $|t-s|$. This is obviously the case if one consider the limit described in the exercise.
d) Let us compute first the following conditional expectation

$$\mathbb{E}(B(\phi(t+h)) \mid B(\phi(t)) = x) \,.$$

Since Brownian motion has stationary independent increments we have

$$\begin{aligned}
\mathbb{E}(B(\phi(t+h)) \mid B(\phi(t)) = x) &= \frac{1}{\sqrt{2\pi(\phi(t+h)-\phi(t))}}\int_{\mathbb{R}} w \exp\left(-\frac{(w-x)^2}{2(\phi(t+h)-\phi(t))}\right) dw \\
&= \frac{1}{\sqrt{2\pi(\phi(t+h)-\phi(t))}}\int_{\mathbb{R}} (x+z) \exp\left(-\frac{z^2}{2(\phi(t+h)-\phi(t))}\right) dz \\
&= x \qquad (9)
\end{aligned}$$

where in the last line we used the change of variable $z = w - x$. To compute the instantaneous mean $a(t,y)$ we evaluate

$$\begin{aligned}
\mathbb{E}(Y(t+h) \mid Y(t) = y) &= \mathbb{E}\left(e^{-\beta(t+h)}\frac{\sigma}{\sqrt{2\beta}}B(\phi(t+h)) \mid e^{-\beta t}\frac{\sigma}{\sqrt{2\beta}}B(\phi(t)) = y\right) \\
&= e^{-\beta(t+h)}\frac{\sigma}{\sqrt{2\beta}}\mathbb{E}\left(B(\phi(t+h)) \mid B(\phi(t)) = \frac{\sqrt{2\beta}}{\sigma}e^{\beta t}y\right)
\end{aligned}$$

Making use of Eq.(9) we arrive to

$$\begin{aligned}
\mathbb{E}(Y(t+h) \mid Y(t) = y) &= e^{-\beta(t+h)}e^{\beta t}y \\
&= y e^{-\beta h}
\end{aligned}$$

from which it is found

$$\mathbb{E}(Y(t+h) - Y(t) \mid Y(t) = y) = y\left(e^{-\beta h} - 1\right) = -\beta y h + o(h)$$

Therefore $a(y,t) = -\beta y$. In a similar way, to evaluate the instantaneous variance $b(t,y)$ we compute

$$\begin{aligned}
\mathbb{E}(Y(t+h)^2 \mid Y(t) = y) &= \mathbb{E}\left(e^{-2\beta(t+h)}\frac{\sigma^2}{2\beta}B(\phi(t+h))^2 \mid e^{-\beta t}\frac{\sigma}{\sqrt{2\beta}}B(\phi(t)) = y\right) \\
&= e^{-2\beta(t+h)}\frac{\sigma^2}{2\beta}\left(\mathbb{E}(B(\phi(t+h))^2 \mid B(\phi(t)) = \frac{\sqrt{2\beta}}{\sigma}e^{\beta t}y\right)
\end{aligned}$$

Using the fact that

$$\begin{aligned}
\mathbb{E}(B(\phi(t+h))^2 \mid B(\phi(t)) = x) &= \frac{1}{\sqrt{2\pi(\phi(t+h)-\phi(t))}}\int_{\mathbb{R}} w^2 \exp\left(-\frac{(w-x)^2}{2(\phi(t+h)-\phi(t))}\right) dw \\
&= \frac{1}{\sqrt{2\pi(\phi(t+h)-\phi(t))}}\int_{\mathbb{R}} (x+z)^2 \exp\left(-\frac{z^2}{2(\phi(t+h)-\phi(t))}\right) dz \\
&= x^2 + \phi(t+h) - \phi(t) \qquad (10)
\end{aligned}$$

and recalling again Eq. (9) we find

$$\begin{aligned}
\mathbb{E}(Y(t+h)^2 \mid Y(t) = y) &= e^{-2\beta(t+h)}\frac{\sigma^2}{2\beta}\left(e^{2\beta t}y^2\frac{2\beta}{\sigma^2} + \phi(t+h) - \phi(t)\right) \\
&= e^{-2\beta(t+h)}\left(e^{2\beta t}y^2 + \frac{\sigma^2}{2\beta}\left(e^{2\beta(t+h)} - e^{2\beta t}\right)\right) \\
&= e^{-2\beta h}y^2 + \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta h}\right)
\end{aligned}$$

We finally have

$$\begin{aligned}
\mathbb{E}((Y(t+h) - Y(t))^2 \mid Y(t) = y) &= \mathbb{E}(Y(t+h)^2 - 2Y(t+h)Y(t) + Y(t)^2 \mid Y(t) = y) \\
&= \mathbb{E}(Y(t+h)^2 \mid Y(t) = y) - 2y\mathbb{E}(Y(t+h) \mid Y(t) = y) + y^2 \\
&= e^{-2\beta h}y^2 + \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta h}\right) - 2y^2 e^{-\beta h} + y^2 \\
&= \sigma^2 h + o(h)\ .
\end{aligned}$$

This yields the instantaneous variance $b(t,y) = \sigma^2$.
e) One immediately obtain that in the limit $\frac{\partial}{\partial t}p(y,t) = 0$ the solution is

$$p(t,y) = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\beta}}}\exp\left(-\frac{y^2}{2\frac{\sigma^2}{2\beta}}\right)\ .$$

That is, in the stationary state, the distribution of Y is $N(0, \frac{\sigma^2}{2\beta})$, as it can be also deduced by taking the limit $t \to \infty$ in a).