Exam Stochastic Processes 2WB08 - January 25, 2008, 14.00-17.00

The number of points that can be obtained per exercise is mentioned between square brackets. The maximum number of points is 40. Good luck!!

Problem 1: Consider a renewal process with distribution $F(\cdot)$ of the times between successive renewals, with mean μ. Let $m(t)$ denote the renewal function of this process.
a) [2 pt.] Argue that $m(t)$ satisfies the following equation:

$$m(t) = F(t) + \int_0^t m(t-x)\mathrm{d}F(x), \quad t \geq 0.$$

b) [2 pt.] Let $\phi(s) := \int_0^\infty \mathrm{e}^{-st}\mathrm{d}F(t)$ and $\mu(s) := \int_0^\infty \mathrm{e}^{-st}\mathrm{d}m(t)$. Prove that $\mu(s) = \phi(s)/(1 - \phi(s))$.
c) [3 pt.] In the sequel, let $F(t)$ be hyperexponential: $F(t) = p(1-\mathrm{e}^{-\lambda_1 t}) + (1-p)(1-\mathrm{e}^{-\lambda_2 t})$, $t \geq 0$, with $0 < p < 1$ and $\lambda_1, \lambda_2 > 0$. Prove that

$$m(t) = \frac{t}{\mu} + B(1 - \mathrm{e}^{-((1-p)\lambda_1 + p\lambda_2)t}), \quad t \geq 0,$$

with B some constant (which you do not have to determine).
d) [3 pt.] Consider the delayed renewal process that starts with a time Y with distribution $G(\cdot)$ until the first renewal, and with subsequently i.i.d. interrenewal times $X_1, X_2, \ldots$ with distribution the hyperexponential distribution $F(\cdot)$ as above. Consider the renewal function $m_D(\cdot)$ for this delayed renewal process. Determine $G(\cdot)$ such that $m_D(t) = t/\mu$.

Problem 2: Let $(X_i)_{i\geq 1}$ be a i.i.d. sequence of random variables with distribution

$$X_i = \begin{cases} 1 & \text{with probability } p \,, \\ 0 & \text{with probability } r \,, \\ -1 & \text{with probability } q \,, \end{cases}$$

where $p, q, r > 0$. Let $(S_n)_{n\geq 1} = S_0 + \sum_{i=1}^n X_i$ be a discrete time random walk starting at $S_0 = m \in \mathbb{N}$ at time zero.
a) [2 pt.] Define $(Y_n)_{n\geq 1} = \left(\frac{q}{p}\right)^{S_n}$. Show that $(Y_n)_{n\geq 1}$ is a martingale and that for any positive integer n one has $\mathbb{E}(Y_n) = \left(\frac{q}{p}\right)^m$.
b) [2 pt.] Let T be the time until the walker reaches either 0 or N for the first time, where N is an integer greater than m. Compute $\mathbb{E}(Y_T)$. If you apply the martingale stopping theorem remember to check that the hypotheses for the applicability of the theorem are satisfied.
c) [3 pt.] Assuming $p \neq q$, compute the probability that, starting from m, the walker reaches 0 before it reaches N.
d) [3 pt.] In the case $p = q$ you may assume that the probability of the previous item is $\mathbb{P}(S_T = 0 \mid S_0 = m) = (N-m)/N$. Now define $Z_n = S_n^2 - 2np$. Prove that $(Z_n)_{n\geq 1}$ is a martingale and show that the expected time until absorption is given by

$$\mathbb{E}(T|S_0 = m) = \frac{m(N-m)}{2p} \,.$$

Problem 3: a) [4 pt.] Consider an insurance company, which receives claims according to a Poisson process with rate λ, claim sizes being exponentially distributed with mean $1/\nu$, and which receives premium at constant rate c. Let the initial capital be A. Assume that the company is allowed to have a negative amount of money, i.e., the company cannot be ruined. Obtain an expression for the Laplace-Stieltjes transform of the capital of the company just after the nth claim. Also obtain the mean of this capital.
b) [2 pt.] Describe a queueing model that gives rise to the same random walk as the above-described insurance model.
c) [4 pt.] Let Y_i denote the ith claim size and X_i the ith interarrival time. Let $EY_i < cEX_i$. Show that

$$\mathbb{P}(\textit{the capital of the company never becomes negative}) \geq 1 - \mathrm{e}^{-\theta A},$$

where θ is such that $E[\mathrm{e}^{\theta(Y_i - cX_i)}] = 1$. Also determine this θ.

Problem 4: Let $B(t)$ be a standard Wiener process. For $\beta > 0$ and $\sigma > 0$, consider the process

$$Y(t) = e^{-\beta t} \frac{\sigma}{\sqrt{2\beta}} B(e^{2\beta t} - 1) \ .$$

a) [1 pt.] Show that the distribution of $Y(t)$ is normal $N\left(0, \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})\right)$.
b) [2 pt.] Is $Y(t)$ a Gaussian process? Compute its covariance $Cov(Y(s), Y(t))$.
c) [2 pt.] Show that in the limit $t, s \to \infty$ with finite $|t - s|$ the process becomes stationary. We recall that a process is said to be stationary if $Y(t_1), \ldots, Y(t_n)$ has the same joint distribution as $Y(t_1 + h), \ldots, Y(t_n + h)$ for all $t_1, \ldots, t_n, h, n$.
d) [3 pt.] Compute the instantaneous mean $a(t, y)$ and the instantaneous variance $b(t, y)$ of $Y(t)$ defined as

$$\mathbb{E}(Y(t+h) - Y(t) \mid Y(t) = y) = a(t, y)\, h + o(h) \ ,$$
$$\mathbb{E}((Y(t+h) - Y(t))^2 \mid Y(t) = y) = b(t, y)\, h + o(h) \ .$$

e) [2 pt.] Find a *stationary* solution for the Kolmogorov forward differential equation

$$\frac{\partial}{\partial t} p(y, t) = -\frac{\partial}{\partial y}(a(t, y) p(y, t)) + \frac{1}{2} \frac{\partial^2}{\partial y^2}(b(t, y) p(y, t)) \ .$$

Hint.: This can be obtained by imposing $\frac{\partial}{\partial t} p(y, t) = 0$.