TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

**Exam Pre-master Probability and Statistics (2DI50) on 23 March 2007, 9.00-12.00**

- At the exam it is allowed to use the 'Statistisch Compendium' and a pocket calculator.
- The exam has 8 exercises with 20 items.
- The answers have to be in ENGLISH. Give not just answers, but also arguments.
- For each item one get two points. In total one can get 40 points.

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**Problem 1:**

In a population of 100 individuals, 5 of them have disease $D_1$, 10 of them have disease $D_2$, and 85 of them have neither disease $D_1$ nor $D_2$. None of the individuals has both diseases.
The same symptom appears among 4 people having disease $D_1$, among 6 people having disease $D_2$ and among 5 people that have neither disease $D_1$ nor $D_2$.

(a) What is the probability that a randomly chosen (from the population of 100 individuals) person showing the symptom has either disease $D_1$ or disease $D_2$ ?
(b) What is the probability that a randomly chosen person showing the symptom has neither disease $D_1$ nor $D_2$.

**Problem 2:**

The amount of bread (measured in hundreds of Kg) sold by a bakery in a day is a random variable $X$ having probability density

$$f_X(x) = \begin{cases} c\,x & 0 \le x < 3 \\ c(6-x) & 3 \le x < 6 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant $c$.
(b) What is the probability that in one day more than 300 Kg (event $A$) are sold? What is the probability that in one day between 150 and 450 Kg (event $B$) are sold?
(c) Are the events $A$ and $B$ independent?

**Problem 3:**

A man with $n$ keys wants to open his door and tries the keys ar random. Exactly one key will open the door.
(a) Suppose that unsuccessful keys are NOT eliminated from the $n$ keys. Determine the probability distribution of $X$, the number of trials until he opens the door. Find the range of $X$ and its expected number of trials.
(b) Suppose that unsuccessful keys are eliminated from the $n$ keys. Determine the probability distribution of the number of trials until he opens the door. Find the expected number of trials.

**Problem 4:**

The length of the time required to complete a college test is found to be normally distributed with mean 50 minutes and standard deviation 12 minutes.
(a) When should the test be terminated if we wish to allow sufficient time for 90% of the students to complete the test?
(b) What proportion of students will finish the test between 30 and 60 minutes?

**Problem 5:**

Wires are manufactured with a nominal length of 20 cm. It is assumed that the length of a wire is a normal gaussian variable with mean $\mu$ and variance $\sigma^2$. A random sample of 25 measurements is taken and it is found a sample mean $\bar{x} = 16$ cm and a sample variance $S^2 = 100$ cm$^2$.
(a) Give a 99% confidence interval for the mean $\mu$.
(b) Discuss whether the nominal length is a plausible value for the mean $\mu$ given the result of part (a).
(c) Give a 99% confidence interval for the variance $\sigma^2$.

**Problem 6:**

Two factories produce light bulbs whose lifetimes (measured in thousands of hours) are exponential random variables with parameter $\lambda_1 = 1$ for the first factory and parameter $\lambda_2 = 2$ for the second factory.
(a) What is the probability that the lifetime of a light bulb produced by the first factory is less then 1000 hours? What is the probability that the lifetime of a light bulb produced by the second factory is less then 1000 hours?

In a batch of 300 light bulbs, 200 have been produced by first factory and 100 have been produced by the second factory. A randomly chosen light bulb is tested and it is found that its lifetime is less than 1000 hours.
(b) What is the probability that this light bulb was produced by the first factory? (Hint: use Bayes formula).

**Problem 7:**

Let $U$ be a random variable which takes values 0, 1 and 4 with probabilities $1-2\theta+\frac{3}{2}\theta^2$, $2\theta(1-\theta)$ and $\frac{1}{2}\theta^2$, respectively.
(a) Show that $\hat{\Theta} = U/2$ is an unbiased estimator for $\theta$.
(b) Compute the mean square error of $\hat{\Theta}$

**Problem 8:**

Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter $p$. The sample mean $\bar{X}$ and sample variance $S^2$ are defined as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad\qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

(a) Show that $S^2$ is an unbiased estimator for the population variance $\sigma^2 = p(1-p)$.
(b) Show that $\bar{X} - S^2$ is an unbiased estimator for $p^2$.
(c) A measure is taken for a random sample of size $n = 80$. This yields a sample mean of $\bar{x} = 0.275$. Compute an approximate 90% confidence interval for $p$, stating clearly any assumption which are necessary for the approximation to be valid.
(d) Suppose now that you would like to estimate $p$ with a precision of $\pm 0.02$, in the sense that a 90% confidence interval should have half-width no greater than 0.02. How large a sample size would be necessary to achieve this precision? Use the sample mean from the previous item involving 80 samples.