Exam Stochastic Processes 2WB08 - March 14, 2008, 14.00-17.00

The number of points that can be obtained per exercise is mentioned between square brackets. The maximum number of points is 40. Good luck!!

Problem 1: Consider a renewal process $\{N(t),\ t \geq 0\}$ with interrenewal times $X_1, X_2, \ldots$ with distribution $F(\cdot)$ and mean μ. Let $m(t)$ denote the renewal function of this process. Let $S_n = \sum_{i=1}^{n} X_i$, with $S_0 = 0$.
a) [3 pt.] Prove that $ES_{N(t)+1} = \mu[m(t) + 1]$.
b) [2 pt.] Use this to show that $lim\ inf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$.
c) [2 pt.] Introduce $\bar{X}_n = \min(X_n, M)$ to prove that $lim\ sup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$.
d) [3 pt.] Suppose that $F(x) = 1 - \mathrm{e}^{-x/\mu}$. What is, for this case, $\lim_{t\to\infty} P(X_{N(t)+1} \leq x)$ and the mean of this limiting distribution?

Problem 2: Let $(X_n)_{n\geq 0}$ be a discrete time Markov chain with countable state space S and transition matrix P. Let $h : S \to \mathbb{R}$ be a function with the properties that

$$i) \qquad \sum_{j\in S} P_{i,j} h(j) = h(i) \quad \forall i \in S$$

$$ii) \qquad \mathbb{E}(|h(X_n)|) < \infty \quad \forall n \geq 0$$

a) [3 pt.] Show that $M_n = h(X_n)$ is a martingale with respect to X_n.
b) [2 pt.] Construct a martingale in the case that property $i)$ is replaced by

$$i^{'}) \qquad \sum_{j\in S} P_{i,j} h(j) = \lambda h(i) \quad \forall i \in S, \quad \lambda \in \mathbb{R}$$

c) [2 pt.] Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain with state space $\mathbb{N}_0$ and transition probabilities

$$\begin{aligned} p(x, x+1) &= p_x \quad \text{for } x \geq 0, \\ p(x, x-1) &= q_x \quad \text{for } x \geq 1, \\ p(x, x) &= r_x \quad \text{for } x \geq 0 \end{aligned}$$

and $p(x, y) = 0$ for all other pairs (x, y); here $p_x, q_x, r_x \geq 0$ with $p_x + q_x + r_x = 1$. Define

$$\phi(j) = \sum_{i=1}^{j-1} \frac{q_1 q_2 \cdots q_i}{p_1 p_2 \cdots p_i}$$

Show that $\phi(X_n)$ constitutes a martingale.
d) [3 pt.] Fix $n, m_1, m_2 \in \mathbb{N}$ such that $m_1 < n < m_2$. Let $\pi(n)$ be the probability that the process hits m_2 before it reaches m_1, having started at n. Show that

$$\pi(n) = \frac{\phi(n) - \phi(m_1)}{\phi(m_2) - \phi(m_1)}.$$

Problem 3: Consider a random walk $\{S_n,\ n \geq 0\}$ with $S_n = \sum_{i=1}^{n} X_i$, $S_0 = 0$. Let $X_i = U_i - V_i$, with U_i, V_i independent, identically distributed and nonnegative random variables; U_i is exponentially distributed with mean a and V_i is exponentially distributed with mean $b > a$. Let $\theta \neq 0$ be such that $E[\mathrm{e}^{\theta X}] = 1$.
a) [2 pt.] Determine θ.
b) [2 pt.] Give the distribution of S_n, as well as the Laplace-Stieltjes transform of this distribution.
c) [2 pt.] Argue that $\{Z_n,\ n = 0, 1, \dots\}$ with $Z_n = \mathrm{e}^{\theta S_n}$ is a martingale.
d) [2 pt.] Let $A > 0, B > 0$ and define the stopping time N as $N = \min\{n : S_n = A \text{ or } S_n = -B\}$. Find an expression for the probability P_A that the random walk reaches A before it reaches $-B$.
e) [2 pt.] Prove that the probability that the random walk ever reaches A is bounded by $\mathrm{e}^{-\theta A}$.

Problem 4: Let $W(t)$ denote a standard Wiener process.
a) [3 pt.] Write down the definition of the process called Brownian bridge. Show that the covariance function of this process is $c(s,t) = \min\{s,t\} - st$ with $0 \leq s, t \leq 1$.
b) [2 pt.] Define $Z_1(t) = W(t) - tW(1)$. Show that $\{Z_1(t) : 0 \leq t \leq 1\}$ is a Brownian bridge.
c) [2 pt.] Define $Z_2(t) = (1-t)W\left(\frac{t}{1-t}\right)$ for $0 \leq t < 1$ and $Z_2(1) = 0$. Show that $\{Z_2(t) : 0 \leq t \leq 1\}$ is a Brownian bridge.
d) [3 pt.] Let $0 < s < t < 1$. Show that the probability that the Brownian bridge has no zero in the interval (s,t) is $\frac{2}{\pi} \arccos \sqrt{\frac{t-s}{t(1-s)}}$. *Hint:* use one of the two previous representations above of the Brownian Bridge in terms of Wiener process. You might assume the knowledge of the arcsine law for Wiener process.