# Exam Stochastic Processes 2WB08 – January 29, 2007, 14.00-17.00

The number of points that can be obtained per exercise is mentioned between sqaure brackets. The maximum number of points is 40. Good luck!!

**Problem 1:**
A light bulb burns for an amount of time having distribution $F(\cdot)$, with Laplace transform $\phi(\cdot)$ of the density, and with mean $\mu$ and second moment $\mu_2$. When the light bulb burns out, it is immediately replaced by another light bulb which has the same life time distribution $F(\cdot)$, etc. Let $m(t)$ be the mean number of replacements of light bulbs upto time $t$.
a) [2 pt.] Show that $m(t) = \sum_{n=1}^{\infty} F_n(t)$, $t \geq 0$, with $F_n(t)$ the $n$-fold convolution of $F(t)$ with itself.
b) [2 pt.] Show that $\mu(s) = \int_0^\infty \mathrm{e}^{-st} dm(t)$ equals $\frac{\phi(s)}{1-\phi(s)}$.
c) [2 pt.] Let $F(\cdot)$ be an Erlang-2 distribution, so having density $(\frac{2}{\mu})^2 t\mathrm{e}^{-2t/\mu}$, $t > 0$. Show that $\mu(s) = \frac{1}{\mu s} - \frac{1}{\mu s+4}$.
d) [2 pt.] Use c) to show that $m(t) = \frac{t}{\mu} - \frac{1}{4}(1 - \mathrm{e}^{-4t/\mu})$, $t \geq 0$.
e) [2 pt] Consider the excess or residual life $Y(t)$ of the light bulb burning at time $t$. Determine an expression for $\lim_{t\to\infty} \mathbb{P}(Y(t) \leq x)$.

**Problem 2:**
Let $(X_n)_{n\geq 0}$ be a sequence of random variables such that $\mathbb{E}(X_n^2) < \infty$ and $\mathbb{E}(X_{n+1} | X_0, \ldots, X_n) = 0$ for all $n \in \mathbb{N}$. Define $S_n = \sum_{i=1}^n X_i$ to be the partial sum (with $S_0 = 0$) and consider the sequence

$$M_n = S_n^2 - \sum_{k=0}^{n-1} \mathbb{E}(X_{k+1}^2 | X_0, \ldots, X_k)$$

a) [2 pt.] Give the definition of a martingale.
b) [3 pt.] Show that $(M_n)_{n\geq 0}$ is a martingale.
c) [2 pt.] Give the definition of a stopping time $T$ with respect to a sequence $(S_n)_{n\geq 0}$.
d) [3 pt.] Let $T_1$ and $T_2$ be two stopping times with respect to a sequence $(S_n)_{n\geq 0}$. Decide whether the following are stopping times with respect to the same sequence: $T_1 + T_2$, $\min(T_1, T_2)$, $\max(T_1, T_2)$.

**Problem 3:**
Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Define, for $a > 0$ and $b < 0$

$$T = \inf\{u \geq 0 : B_u \in \{a, b\}\} .$$

a) [2 pt.] Show that $B_t$ is a martingale.
b) [3 pt.] By applying the stopping theorem to the martingale $(B_t)_{t\geq 0}$ and the stopping time $T$ show that

$$\mathbb{P}(B_T = a) = \frac{-b}{a-b} .$$

Define now

$$M_t = \int_0^t B_u du - \frac{1}{3} B_t^3 .$$

c) [3 pt.] Show that $(M_t)_{t\geq 0}$ is a martingale.
d) [2 pt.] Deduce that the expected area under the path of $B_t$ until it first reaches one of the levels $a$ or $b$ is

$$-\frac{1}{3}ab(a+b)\,.$$

Hint: apply once more the stopping theorem, this time to the martingale $(M_t)_{t\geq 0}$.

**Problem 4:**
Consider the $G/G/1$ queue, with i.i.d. interarrival times $X_i$ and i.i.d. service times $Y_i$, $i = 1, 2, \ldots$, and $EY_i < EX_i$. Let $D_n$ denote the delay (waiting time) of the $n$-th arriving customer, $n = 1, 2, \ldots$. Let $D_1 = 0$.
a) [2 pt.] Argue that $D_{n+1} = \max(0, D_n + Y_n - X_{n+1})$, $n = 1, 2, \ldots$.
b) [3 pt.] Let $U_n = Y_n - X_{n+1}$ and $S_n = \sum_{j=1}^{n} U_j$, $n = 1, 2, \ldots$. Show that $\mathbb{P}(D_{n+1} \geq c) = \mathbb{P}(\max(0, S_1, S_2, \ldots, S_n) \geq c)$.
c) [5 pt.] Use Spitzer's identity, viz., $\mathbb{E}[\max(0, S_1, S_2, \ldots, S_n)] = \sum_{k=1}^{n} \frac{1}{k}\mathbb{E}[\max(0, S_k)]$, to show the following if the $X_i$ are exponentially distributed with mean $1/\lambda$ and the $Y_i$ are exponentially distributed with mean $1/\nu$:
$\mathbb{E}D_{n+1} = \sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{k-1} \frac{k-i}{\nu} \binom{k+i-1}{i} (\frac{\lambda}{\lambda+\nu})^k (\frac{\nu}{\lambda+\nu})^i$.
*Hint: use either the approach of the book, or the one from class.*