Exam Stochastic Processes 2WB08 – January 29, 2007

Problem 1:
A light bulb burns for an amount of time having distribution $F(\cdot)$, with Laplace transform $\phi(\cdot)$ of the density, and with mean μ and second moment μ_2. When the light bulb burns out, it is immediately replaced by another light bulb which has the same life time distribution $F(\cdot)$, etc. Let $m(t)$ be the mean number of replacements of light bulbs upto time t.
a) [2 pt.] Show that $m(t) = \sum_{n=1}^{\infty} F_n(t)$, $t \geq 0$, with $F_n(t)$ the n-fold convolution of $F(t)$ with itself.
b) [2 pt.] Show that $\mu(s) = \int_0^\infty \mathrm{e}^{-st} dm(t)$ equals $\frac{\phi(s)}{1-\phi(s)}$.
c) [2 pt.] Let $F(\cdot)$ be an Erlang-2 distribution, so having density $(\frac{2}{\mu})^2 t\mathrm{e}^{-2t/\mu}$, $t > 0$. Show that $\mu(s) = \frac{1}{\mu s} - \frac{1}{\mu s+4}$.
d) [2 pt.] Use c) to show that $m(t) = \frac{t}{\mu} - \frac{1}{4}(1 - \mathrm{e}^{-4t/\mu})$, $t \geq 0$.
e) [2 pt] Consider the excess or residual life $Y(t)$ of the light bulb burning at time t. Determine an expression for $\lim_{t\to\infty} \mathbb{P}(Y(t) \leq x)$.

Solution Problem 1:
a) Let $N(t)$ denote the number of replacements up to time t. Then $\{N(t), t \geq 0\}$ is a renewal process. The inter-arrival times $X_1, X_2, \ldots$ are i.i.d. with distribution $F(\cdot)$, then the partial sum $S_n = \sum_{i=1}^n X_i$ is distributed like $F_n(\cdot)$, the n-fold convolution of $F(\cdot)$ with itself. We have

$$m(t) = \mathbb{E}(N(t)) = \sum_{n=1}^{\infty} \mathbb{P}(N(t) \geq n) = \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq t) = \sum_{n=1}^{\infty} F_n(t)$$

b) You can either condition on the first arrival time or you might compute:

$$\mu(s) = \int_0^\infty \mathrm{e}^{-st} dm(t) = \int_0^\infty \mathrm{e}^{-st} d[\sum_{n=1}^{\infty} F_n(t)]$$

Exchanging integration and sum

$$\mu(s) = \sum_{n=1}^{\infty} (\phi(s))^n = \frac{\phi(s)}{1 - \phi(s)}$$

c) Using integration by parts one finds

$$\phi(s) = \int_0^\infty \left(\frac{2}{\mu}\right)^2 t\mathrm{e}^{-2t/\mu} e^{-st} dt = \left(\frac{2}{2 + \mu s}\right)^2$$

From the result b) one has

$$\mu(s) = \frac{1}{\mu s} - \frac{1}{\mu s + 4}$$

d) From the proposed $m(t)$ we have $dm(t) = m'(t)dt = \left(\frac{1}{\mu} - \frac{1}{\mu} e^{-4t/\mu}\right) dt$ so that $\mu(s) = \int_0^\infty \mathrm{e}^{-st} dm(t)$ equals $\frac{1}{\mu s} - \frac{1}{\mu s+4}$ as it should. This implies that $m(t) = \frac{t}{\mu} + \frac{1}{4} e^{-4t/\mu} + C$

where C is a constant to be determined. On the other hand we know that $m(0) = 0$ which yields $C = -\frac{1}{4}$.
e)

$$\lim_{t\to\infty}\mathbb{P}(Y(t) \le x) = \int_0^x \bar{F}(y)dy/\mu$$

For a proof see Prop. 3.4.5 on the Ross book.

Problem 2:
Let $(X_n)_{n\ge 0}$ be a sequence of random variables such that $\mathbb{E}(X_n^2) < \infty$ and $\mathbb{E}(X_{n+1}|\, X_0,\ldots,X_n) = 0$ for all $n \in \mathbb{N}$. Define $S_n = \sum_{i=1}^n X_i$ to be the partial sum (with $S_0 = 0$) and consider the sequence

$$M_n = S_n^2 - \sum_{k=0}^{n-1} \mathbb{E}(X_{k+1}^2|\, X_0,\ldots,X_k)$$

a) [2 pt.] Give the definition of a martingale.
b) [3 pt.] Show that $(M_n)_{n\ge 0}$ is a martingale.
c) [2 pt.] Give the definition of a stopping time T with respect to a sequence $(S_n)_{n\ge 0}$.
d) [3 pt.] Let T_1 and T_2 be two stopping times with respect to a sequence $(S_n)_{n\ge 0}$. Decide whether the following are stopping times with respect to the same sequence: $T_1 + T_2$, $\min(T_1, T_2)$, $\max(T_1, T_2)$.

Solution problem 2:
a) A sequence $(M_n)_{n\ge 0}$ of random variables is called a martingale if for all n the following holds:

$$\mathbb{E}(|M_n|) < \infty \qquad \text{and} \qquad \mathbb{E}(M_{n+1}|\, M_0,\ldots,M_n) = M_n$$

b) We show that $(M_n)_{n\ge 0}$ is a martingale with respect to $(X_n)_{n\ge 0}$. Note that

$$\mathbb{E}(|M_n|) \le \mathbb{E}(S_n^2) + \sum_{k=0}^{n-1} \mathbb{E}(\mathbb{E}(X_{k+1}^2|\, X_0,\ldots,X_k))\,.$$

From the properties of conditional expectation one has $\mathbb{E}(\mathbb{E}(X_{k+1}^2|\, X_0,\ldots,X_k)) = \mathbb{E}(X_{k+1}^2)$. The integrability condition $\mathbb{E}(|M_n|) < \infty$ for all $n \in \mathbb{N}$ then follows from the hypothesis $\mathbb{E}(X_n^2) < \infty$. Furthermore,

$$\begin{aligned}
\mathbb{E}(M_{n+1}|\, X_0,\ldots,X_n) &= \mathbb{E}\Big(\Big[S_{n+1}^2 - \sum_{k=0}^{n} \mathbb{E}(X_{k+1}^2|\, X_0,\ldots,X_k)\Big]\,|\, X_0,\ldots,X_n\Big) \\
&= \mathbb{E}(S_n^2 + 2S_nX_{n+1} + X_{n+1}^2|\, X_0,\ldots,X_n) + \\
&\quad - \sum_{k=0}^{n} \mathbb{E}(\mathbb{E}(X_{k+1}^2|X_0,\ldots,X_k)|\, X_0,\ldots,X_n) \\
&= S_n^2 + \mathbb{E}(X_{n+1}^2|\, X_0,\ldots,X_n) - \sum_{k=0}^{n} \mathbb{E}(X_{k+1}^2|\, X_0,\ldots,X_k) \\
&= M_n
\end{aligned}$$

where in the second to third line we have used the hypothesis $\mathbb{E}(X_{n+1} | X_0, \dots, X_n) = 0$ and the properties of conditional expectations. Hence, $(M_n)_{n\geq 0}$ is a martingale with respect to $(X_n)_{n\geq 0}$, and thus, it is a martingale.
c) A random variable T with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$ is called a stopping time with respect to $(S_n)_{n\geq 0}$ if $\{T = n\} \in \sigma(S_0, \dots, S_n)$ for all $n \geq 0$.
d) They are all stopping time, as it is shown by observing that

$$\{T_1 + T_2 = n\} = \bigcup_{k=0}^{n} [\{T_1 = k\} \cap \{T_2 = n - k\}]$$

$$\{\min(T_1, T_2) \leq n\} = \{T_1 \leq n\} \cup \{T_2 \leq n\}$$
$$\{\max(T_1, T_2) \leq n\} = \{T_1 \leq n\} \cap \{T_2 \leq n\}$$

Problem 3:
Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Define, for $a > 0$ and $b < 0$

$$T = \inf\{u \geq 0 : B_u \in \{a, b\}\} .$$

a) [2 pt.] Show that B_t is a martingale.
b) [3 pt.] By applying the stopping theorem to the martingale $(B_t)_{t\geq 0}$ and the stopping time T show that

$$\mathbb{P}(B_T = a) = \frac{-b}{a - b} .$$

Define now

$$M_t = \int_0^t B_u du - \frac{1}{3} B_t^3 .$$

c) [3 pt.] Show that $(M_t)_{t\geq 0}$ is a martingale.
d) [2 pt.] Deduce that the expected area under the path of B_t until it first reaches one of the levels a or b is

$$-\frac{1}{3} ab(a + b) .$$

Hint: apply once more the stopping theorem, this time to the martingale $(M_t)_{t\geq 0}$.

Solution problem 3:
a) Write B_t as $B_s + (B_t - B_s)$ and utilize independent increments:

$$\begin{aligned} \mathbb{E}(B_t | B_u, 0 \leq u \leq s) &= \mathbb{E}(B_s | B_u, 0 \leq u \leq s) + \mathbb{E}(B_t - B_s | B_u, 0 \leq u \leq s) \\ &= B_s + \mathbb{E}(B_t - B_s) \\ &= B_s \end{aligned} \tag{1}$$

b) The application of the stopping theorem yields $\mathbb{E}(B_T) = \mathbb{E}(B_0) = 0$. We have $\mathbb{E}(B_T) = a\mathbb{P}(B_T = a) + b\mathbb{P}(B_T = b)$ and $\mathbb{P}(B_T = a) + \mathbb{P}(B_T = b) = 1$. From this we deduce that

$$\mathbb{P}(B_T = a) = \frac{-b}{a - b} . \tag{2}$$

c) To show that $(M_t)_{t\geq 0}$ is a martingale we prove that $\forall s < t$

$$\mathbb{E}(M_t|\, B_u, 0 \leq u \leq s) = M_s \tag{3}$$

We have

$$\begin{aligned}
\mathbb{E}\left(\int_0^t B_u du|\, B_u, 0 \leq u \leq s\right) &= \mathbb{E}\left(\int_0^s B_u du|\, B_u, 0 \leq u \leq s\right) + \mathbb{E}\left(\int_s^t B_u du|\, B_u, 0 \leq u \leq s\right) \\
&= \int_0^s B_u du + \mathbb{E}\left(\int_s^t (B_u - B_s + B_s) du|\, B_u, 0 \leq u \leq s\right) \\
&= \int_0^s B_u du + \int_s^t \mathbb{E}(B_u - B_s) du + B_s(t-s) \\
&= \int_0^s B_u du + B_s(t-s) \qquad (4)
\end{aligned}$$

where in the third line we use the property of stationary independent increments for Brownian motion. The same property gives also that

$$\mathbb{E}(B_t^3|\, B_u, 0 \leq u \leq s) = B_s^3 + 3B_s(t-s) \tag{5}$$

Combining together (4) and (5) and using the definiton of M_t proves (3).
d) From the application of the stopping theorem to the martingale $(M_t)_{t\geq 0}$ it follows:

$$\mathbb{E}\left(\int_0^T B_u du - \frac{1}{3} B_T^3\right) = 0$$

Hence the required area A has mean

$$\mathbb{E}(A) = \mathbb{E}\left(\int_0^T B_u du\right) = \frac{1}{3}\mathbb{E}(B_T^3) = \frac{1}{3}a^3\left(\frac{-b}{a-b}\right) + \frac{1}{3}b^3\left(\frac{a}{a-b}\right) = -\frac{1}{3}ab(a+b)$$

where we made use of the previous result (2) to evaluate the expectation.

Problem 4:
Consider the $G/G/1$ queue, with i.i.d. interarrival times X_i and i.i.d. service times Y_i, $i = 1, 2, \ldots$, and $EY_i < EX_i$. Let D_n denote the delay (waiting time) of the n-th arriving customer, $n = 1, 2, \ldots$. Let $D_1 = 0$.
a) [2 pt.] Argue that $D_{n+1} = \max(0, D_n + Y_n - X_{n+1})$, $n = 1, 2, \ldots$.
b) [3 pt.] Let $U_n = Y_n - X_{n+1}$ and $S_n = \sum_{j=1}^n U_j$, $n = 1, 2, \ldots$. Show that $\mathbb{P}(D_{n+1} > c) = \mathbb{P}(\max(0, S_1, S_2, \ldots, S_n) > c)$.
c) [5 pt.] Use Spitzer's identity, viz., $\mathbb{E}[\max(0, S_1, S_2, \ldots, S_n)] = \sum_{k=1}^n \frac{1}{k}\mathbb{E}[\max(0, S_k)]$, to show the following if the X_i are exponentially distributed with mean $1/\lambda$ and the Y_i are exponentially distributed with mean $1/\nu$:
$\mathbb{E}D_{n+1} = \sum_{k=1}^n \frac{1}{k} \sum_{i=0}^{k-1} \frac{k-i}{\nu} \binom{k+i-1}{i} (\frac{\lambda}{\lambda+\nu})^k (\frac{\nu}{\lambda+\nu})^i$.

Solution Problem 4:
a) The waiting time of the $(n+1)$-th customer is that of the n-th customer plus the service

time of this job minus the interarrival time. Of course the delay can not be negative, which implies $D_{n+1} = \max(0, D_n + Y_n - X_{n+1})$.

b) By iterating the previous relation and using $D_1 = 0$ we have

$$D_{n+1} = \max(0, U_n, U_n + U_{n-1}, U_n + U_{n-1} + U_{n-2}, \ldots, U_n + U_{n-1} + \ldots + U_1)$$

Now

$$\begin{aligned}\mathbb{P}(D_{n+1} \geq c) &= \mathbb{P}(\max(0, U_n, U_n + U_{n-1}, U_n + U_{n-1} + U_{n-2}, \ldots, U_n + U_{n-1} + \ldots + U_1) \geq c) \\ &= \mathbb{P}(\max(0, U_1, U_1 + U_2, U_1 + U_2 + U_3, \ldots, U_1 + U_2 + \ldots + U_n) \geq c) \\ &= \mathbb{P}(\max(0, S_1, S_2, \ldots, S_n) > c)\end{aligned}$$

where in the second line we used duality.

c) See Example 7.1B in the Ross book. Alternatively (as in class), observe that $\mathbb{E}[\max(0, S_k)] = \mathbb{E}[\sum_{i=1}^k Y_i - \sum_{i=1}^k X_{i+1}(\sum_{i=1}^k Y_i > \sum_{i=1}^k X_{i+1})]$ can be handled by observing that the latter difference, if positive, is with probability $\binom{k+i-1}{i}(\frac{\lambda}{\lambda+\nu})^k(\frac{\nu}{\lambda+\nu})^i$ the sum of $k - i$ random variables which are $\exp(\nu)$ distributed.