Course 2S620, Lecture 2
Ferromagnets

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1 Ferromagnets on $\mathbb{Z}^D$: general results

In this lecture we will discuss phase transitions. One of the major success of equilibrium statistical mechanics is its capability to explain the phenomenon of phase transitions in terms of symmetry breaking. The simplest setting to introduce phase transition is by considering *ferromagnetic* models on a lattice.

We start by defining the model. Let $\Lambda \subseteq \mathbb{Z}^D$ be a finite subset of the D dimensional lattice. We will identify Λ with the hypercube $\{1, 2, \ldots, N\}^D$. A site $i \in \Lambda$ will then have coordinates $i = (i_1, i_2, \ldots, i_D)$, where each $i_d \in \{1, \ldots, N\}$ for $d = 1, \ldots, D$. The total number of sites is $|\Lambda| = N^D$.

To each site there is assigned a dichotomic spin variable σ_i which takes the values 1 (spin-up) or -1 (spin down). The configuration space is the set Ω_Λ of all the sequences $\sigma = \{\sigma_i\, , i \in \Lambda\}$. Thus $\Omega_\Lambda = \{-1, 1\}^\Lambda$ and the cardinality of the configuration space is $2^{|\Lambda|} = 2^{N^D}$.

The spin random variables interact among themselves. The interaction is given by the following *Hamiltonian* function $H : \Omega_\Lambda \to \mathbb{R}$

$$H_\Lambda(\sigma) = -\frac{1}{2} \sum_{i,j \in \Lambda} J_{i,j} \sigma_i \sigma_j - h \sum_{i \in \Lambda} \sigma_i \tag{1.1}$$

We assume the coupling interaction $J_{i,j}$ to be a non-negative (ferromagnetic), symmetric, translation invariant function on $\mathbb{Z}^D$, namely $J_{i,j} = J(i-j) = J(j-i) \geq 0$. The factor $1/2$ in the Hamiltonian is just a convention and is included because of the double counting in the sum for $i \neq j$, which is due to the symmetry of $J_{i,j}$. Each $J_{i,j}$ tunes the interaction between the spin i and j. An interaction is said to have finite range if there exist $R \in \mathbb{N}$ such that $J(k)$ equals 0 for all k with $||k|| > R$ (R is called

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the range of the interaction). The main example we will consider is the **Ising** model. In this case, for a given number $J > 0$ we define $J(i-j)$ to be J if $||i-j|| = 1$ and zero otherwise. In other words this interaction couples only *nearest neighbor* sites and it has range 1.

The *external field* $h \in \mathbb{R}$ in the Hamiltonian (1.1) represents the strength of the applied magnetic field. Note that the term $h\sum_i \sigma_i$ breaks the symmetry $\sigma \to -\sigma$ in the Hamiltonian.

The apriori distribution of the spin is the uniform product measure on Ω_Λ, that is $\mathbb{P}(\sigma) = \frac{1}{2^{|\Lambda|}}$ for all $\sigma \in \Omega_\Lambda$. The thermodynamic property of the model will be deduced by considering the thermodynamic limit of the finite volume Gibbs canonical measure

$$\mathbb{P}_{\Lambda,\beta,h}(\sigma) = \frac{e^{-\beta H_\Lambda(\sigma)}}{Z_\Lambda(\beta,h)}\mathbb{P}(\sigma) \tag{1.2}$$

where $Z_\Lambda(\beta,h)$ is the partition function

$$Z_\Lambda(\beta,h) = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H_\Lambda(\sigma)}\mathbb{P}(\sigma)$$

Most of the information about the system will be contained in the thermodynamic limit of the free energy, whose definition we recall:

$$f(\beta,h) = \lim_{\Lambda\nearrow\mathbb{Z}^D} f_\Lambda(\beta,h) \tag{1.3}$$

where the finite volume free energy per particle is given by

$$f_\Lambda(\beta,h) = -\frac{1}{|\Lambda|\beta}\log Z_\Lambda(\beta,h) \tag{1.4}$$

The thermodynamic behavior of the model will be a function of two control parameters, which have a direct physical meaning: the inverse temperature $\beta = 1/T$ and the external field h. A phase transition is characterized by an *order parameter*, i.e. a number (or a function in more general cases) which change its value (or its properties in more general cases) as one of the control parameter crosses a *critical value*. In the case of our setting (ferromagnets) the appropriate order parameter is the *spontaneous magnetization*, which we are going to define now.

Let $\omega_{\Lambda,\beta,h}(\cdot)$ denote expectation w.r.t. the finite volume Gibbs state in formula (1.2), i.e. for a generic bounded function $g(\sigma) : \Omega_\Lambda \to \mathbb{R}$,

$$\omega_{\Lambda,\beta,h}(g(\sigma)) = \sum_{\sigma\in\Omega_\Lambda} g(\sigma)\mathbb{P}_{\Lambda,\beta,h}(\sigma)$$

and let $m_\Lambda(\sigma)$ be the function which measures the sample magnetization per site in the volume Λ

$$m_\Lambda(\sigma) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\sigma_i$$

We define the *specific magnetization* $m(\beta, h)$ as the thermodynamic limit of the expected value of the sample magnetization per site with respect to the Gibbs measure, namely

$$m(\beta, h) = \lim_{\Lambda \nearrow \mathbb{Z}^D} m_\Lambda(\beta, h) \tag{1.5}$$

where

$$m_\Lambda(\beta, h) = \omega_{\Lambda,\beta,h}(m_\Lambda(\sigma)) \tag{1.6}$$

Notice that the following relations is an immediate consequence of Eq. (1.4) and (1.6):

$$m_\Lambda(\beta, h) = -\frac{\partial f_\Lambda(\beta, h)}{\partial h} \tag{1.7}$$

Standard concavity arguments and the Lebesgue dominated convergence theorem can be used to show that an analogous relation holds in the thermodynamic limit:

$$m(\beta, h) = -\frac{\partial f(\beta, h)}{\partial h} \tag{1.8}$$

The specific magnetization $m(\beta, h)$ has the following behavior as the values of the field and temperature are varied. Observe that by the symmetry $\sigma \to -\sigma$ it follows that $m_\Lambda(\beta, 0) = 0$ and thus $m(\beta, 0) = 0$, while for $h > 0$ (respectively $h < 0$) we have $m(\beta, h) > 0$ (respectively $m(\beta, h) < 0$). However there exist a critical value of the β, called the *critical inverse temperature* and denoted by β_c, which has the following properties:

- if $\beta < \beta_c$ then $\lim_{h\to 0^+} m(\beta, h) = \lim_{h\to 0^-} m(\beta, h) = 0$
- if $\beta > \beta_c$ then $\lim_{h\to 0^+} m(\beta, h) = m^* > 0$ and $\lim_{h\to 0^-} m(\beta, h) = -m^* < 0$

We call m^* the *spontaneous magnetization.* It follows from above that in ferromagnets the spontaneous magnetization is the order parameter. Indeed, as a function of the temperature, it is zero above the critical temperature $1/\beta_c$ and is different from zero below the critical temperature, reaching the value 1 as $T \to 0$.

Besides the order parameter, also *correlations* in the Ising model are related to the phase transition. This can be understood from the following heuristic discussion. Fix $h = 0$. At infinite temperature $\beta = 0$, the Gibb state is the uniform product state, with respect to wich the spin are independent and thus uncorrelated. At small but non zero β, spin begin to be positively correlated with their neighbours because of the attractive interaction in the Hamiltonian. Since $\omega_{\beta,0}(\sigma_i) = 0$ for each i, the covariance between spin will coincide with $\omega_{\beta,0}(\sigma_i\sigma_j)$. The correlations turns out to have an exponential decay when the Euclidean distance is large

$$\omega_{\beta,0}(\sigma_i\sigma_j) \sim \exp\left[-\frac{||i-j||}{\xi(\beta, 0)}\right] \qquad \text{as} \quad ||i-j|| \to \infty$$

This relation defines the *correlation length* $\xi(\beta, 0)$, which is a rough measure of the distance over which correlations between spins are significant. As β increase, $\xi(\beta, 0)$

increase, and correlations begin to extend over larger and larger distances. These correlations take the forms of spin fluctuations, which are islands of a few spin each that mostly point in the same direction. When $\beta = \beta_c$ the correlation length is infinite, and this reflected in the fact that $\omega_{\beta_c,0}(\sigma_i\sigma_j)$ decays as a power law:

$$\omega_{\beta_c,0}(\sigma_i\sigma_j) \sim ||i-j||^{-z} \qquad \text{as} \quad ||i-j|| \to \infty$$

where z is some positive number ($z = 1/4$ for the Ising model in $D = 2$). The infinite correlation length at $\beta = \beta_c$ is related to the behavior of the *magnetic susceptibility* $\chi(\beta, 0)$ at zero field, where

$$\chi(\beta, h) = \frac{\partial m(\beta, h)}{\partial h} \tag{1.9}$$

On a finite size system we have

$$\chi_\Lambda(\beta, 0) = \frac{\partial m_\Lambda(\beta, h)}{\partial h} = \frac{\beta}{|\Lambda|}\sum_{i,j\in\Lambda} \omega_{\Lambda,\beta,0}(\sigma_i\sigma_j) = \beta \sum_{j\in\Lambda} \omega_{\Lambda,\beta,0}(\sigma_0\sigma_j)$$

where in the last equality we used translational invariance. Going to the infinite volume limit, we see that for $0 < \beta < \beta_c$, correlations decays exponentially and the susceptibility is finite. By contrast, at $\beta = \beta_c$ correlations decays as power law and the susceptibility is infinite. From the definition of the susceptibility we also see that it is related to the second derivative of the free energy function. This is why the phase transition in ferromagnets is called a second order phase transition. According to a general classification scheme introduced by Erenfhest, a divergence in the k^{th} derivative of the free energy with respect to the one of the control parameters signals a transition of the k^{th} order.

In the Ising model the value of the critical inverse temperature β_c depends on the coupling constant J and - more importantly - on the dimension D of the lattice. We will prove that if $D = 1$ then $\beta = \infty$ and thus spontaneous magnetization does not occur at any finite temperature. By contrast, for any $D \geq 2$, β_c is finite. For the Ising model on $\mathbb{Z}^2$ with zero external fields, Onsanger found the exact value of the free energy in 1944. This calculation, one of the most famous in mathematical physics and for which he won the Nobel prize, involved the transfer matrix formalism (a simple application of this will be given in $D = 1$). Five years later, he announced without proof the exact value of β_c and the spontaneous magnetization, which are given by

$$\sinh 2\beta_c J = 1$$

$$m^* = [1 - \sinh(2\beta J)^{-4}]^{1/8}$$

and were later confirmed to be correct. There is not yet an exact for the Ising model in $D = 3$. However the existence of non zero spontaneous magnetization and the occurrence of a phase transition is established via Peierls argument (see Ellis book, Ch V.5).

2 Curie-Weiss model

The so called *mean-field* theory of ferromagnets is an exactly solvable model which is a prototype to show the existence of a phase transition. The analysis of the model, which is called the *Curie-Weiss* model, involves an interesting application of large deviation theory. The model is obtained by considering an interaction between all possible couples of $N \in \mathbb{N}$ spins. Since spatial distance does not play a role anymore, we can completely forget the geometry of the underling lattice. The subset Λ can be simply taken to be the set of point $\Lambda = \{1, 2, \ldots, N\}$ and its cardinality will then be $|\Lambda| = N$. The sample space $\Omega_N = \{-1, +1\}^N$ will have cardinality $|\Omega_N| = 2^N$. In order for the Hamiltonian function to be extensive in the volume we put $J_{i,j} = 1/N$ for all couples $(i, j) \in \Lambda \times \Lambda$. The Hamiltonian the reads:

$$H_N(\sigma) = -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \tag{2.1}$$

What makes easily solvable the model is the fact that the Hamiltonian can be immediately rewritten in terms of the sample magnetization

$$m_N(\sigma) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \tag{2.2}$$

Indeed we have

$$H_N(\sigma) = -\frac{N}{2} m_N(\sigma)^2 - N h m_N(\sigma) \tag{2.3}$$

We will be interested in studying the thermodynamic limit of the free energy. To lighten the notation we also define the finite volume *pressure*

$$p_N(\beta, h) = \frac{1}{N} \log Z(\beta, h) = \log \sum_{\sigma \in \Omega_N} e^{-\beta H(\sigma)} - N \log 2 \tag{2.4}$$

The pressure is related to the free energy by the formula

$$-\beta f_N(\beta, h) = p_N(\beta, h)$$

2.1 Thermodynamic limit

The first problem is to show the existence of the thermodynamic limit of the free energy (or equivalently of the pressure). To this aim we introduce an *interpolation method*, which is a standard tool in statistical mechanics.

Consider the splitting of a system of size N in two smaller subsystems having N_1 and N_2 sites,respectively, with $N = N_1 + N_2$. For the extensive pressure $P_N(\beta, h) = N p_N(\beta, h)$, we then have the following proposition

Proposition 2.5. *The extensive pressure of the Curie-Weiss model is a subadditive sequence in N, that is*

$$P_N(\beta, h) \leq P_{N_1}(\beta, h) + P_{N_2}(\beta, h) \tag{2.6}$$

Proof. The proof is very simple. For a given spin configuration $\sigma = (\sigma_1, \dots, \sigma_N)$ let us denote in the most natural way by $\sigma' = (\sigma_1, \dots, \sigma_{N_1})$ the spin variable for the subsystem of size N_1 and by $\sigma'' = (\sigma_{N_1+1}, \dots, \sigma_N)$ the N_2 spin variables for the second subsystem. Obviously we have $\sigma = \sigma' \cup \sigma''$.
For a parameter $t \in [0,1]$ consider the following interpolating Hamiltonian

$$H_{N,N_1,N_2}(\sigma,t) = tH_N(\sigma) + (1-t)[H_{N_1}(\sigma') + H_{N_2}(\sigma'')] \tag{2.7}$$

and the interpolating pressure:

$$P_N(\beta,h,t) = \log\left(\sum_{\sigma\in\Omega_N} e^{-\beta H(\sigma,t)}\right) \tag{2.8}$$

The following boundary conditions holds:

$$P_N(\beta,h,1) = \log\left(\sum_{\sigma\in\Omega_N} e^{-\beta H_N(\sigma)}\right) = P_N(\beta,h) \tag{2.9}$$

$$\begin{aligned} P_N(\beta,h,0) &= \log\left(\sum_{\sigma\in\Omega_N} e^{-\beta[H_{N_1}(\sigma')+H_{N_2}(\sigma'')]}\right) \\ &= \log\left(\sum_{\sigma'\in\Omega_{N_1}} e^{-\beta H_{N_1}(\sigma')} \sum_{\sigma''\in\Omega_{N_2}} e^{-\beta H_{N_2}(\sigma'')}\right) \\ &= P_{N_1}(\beta,h) + P_{N_2}(\beta,h) \end{aligned} \tag{2.10}$$

The interpolating pressure is monotone in t. To show this compute the t derivative

$$\frac{d}{dt}P_N(\beta,h,t) = -\beta\,\omega_{N,\beta,h,t}\left(H_N(\sigma) - H_{N_1}(\sigma') - H_{N_2}(\sigma'')\right) \tag{2.11}$$

where $\omega_{N,\beta,h,t}(\cdot)$ denotes expectation with respect to the t deformed state:

$$\omega_{N,\beta,h,t}(g) = \sum_{\sigma\in\Omega_N} g(\sigma)\frac{e^{-\beta H_{N,N_1,N_2}(\sigma,t)}}{Z_{N,N_1,N_2}(\beta,h,t)}\mathbb{P}(\sigma)$$

Observe that

$$\begin{aligned} H_N(\sigma) - H_{N_1}(\sigma') - H_{N_2}(\sigma'') &= -\frac{N}{2}\left(m_N(\sigma)^2 - \frac{N_1}{N}m_{N_1}(\sigma')^2 - \frac{N_2}{N}m_{N_2}(\sigma'')^2\right) \\ &\quad -Nh\left(m_N(\sigma) - \frac{N_1}{N}m_{N_1}(\sigma') - \frac{N_2}{N}m_{N_2}(\sigma'')\right) \end{aligned}$$

where the sample magnetization for the two subsystem are defined by adapting the definition (2.2):

$$m_{N_1}(\sigma') = \frac{1}{N_1}\sum_{i=1}^{N_1}\sigma_i \tag{2.12}$$

$$m_{N_2}(\sigma'') = \frac{1}{N_2} \sum_{i=N_1+1}^{N} \sigma_i \tag{2.13}$$

We then have, combining (2.2),(2.12) and (2.13)

$$m_N(\sigma) - \frac{N_1}{N} m_{N_1}(\sigma') - \frac{N_2}{N} m_{N_2}(\sigma'') = 0 \tag{2.14}$$

and, since the mapping $m \to m^2$ is convex, we have also the general bound, holding for all spin configuration σ

$$m_N(\sigma)^2 - \frac{N_1}{N} m_{N_1}(\sigma')^2 - \frac{N_2}{N} m_{N_2}(\sigma'')^2 \leq 0 \tag{2.15}$$

Putting together (2.11) (2.14) and (2.15), we find that

$$H_N(\sigma) - H_{N_1}(\sigma') - H_{N_2}(\sigma'') \geq 0$$

which implies that

$$\frac{d}{dt} P_N(\beta, h, t) \leq 0$$

since $\omega_{N,\beta,h,t}(g) \geq 0$ if $g \geq 0$. The subadditivity property (2.6) follows then from a straightforward integration of previous expression and application of the fundamental theorem of calculus. □

The following theorem is then an immediate consequence of the previous proposition

Theorem 2.16. *The thermodynamic limit of the pressure (and thus of the free energy) per particle exists and is finite for all β and h.*

Proof. Since $P_N(\beta, h)$ is a subadditive sequence then the limit $\lim_{N\to\infty} \frac{P_N(\beta,h)}{N}$ exists. To show that it is finite it is enough to prove that the sequence $p_N(\beta, h)$ is bounded from below. This follows from because

$$\frac{1}{N} \log\Big(\sum_{\sigma \in \Omega_N} e^{\beta N (\frac{m_N^2(\sigma)}{2} + h m_N(\sigma))} \Big) \geq \frac{1}{N} \log e^{\beta N (\frac{1}{2} + |h|)} = \beta\Big(\frac{1}{2} + |h|\Big)$$

□

2.2 Large deviation solution

The explicit computation of the value of pressure in the thermodynamic limit is a nice application of large deviation theory. Let $Q_N(dz)$ denote the distribution of the sum

$$m_N(\sigma) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i$$

with respect to the product measure $\mathbb{P}(\sigma)$. Then the partition function cab be written as

$$Z_N(\beta,h)=\int_{\mathbb{R}}\exp\left[N(\frac{\beta}{2}z^2+\beta hz)\right]Q_N(dz)$$

By Cramer's theorem, the distributions $\{Q_N;N=1,2,\ldots\}$ have a large deviation property with rate N and rate function

$$I(z)=\sup_{t\in\mathbb{R}}\{tz-\log(\mathbb{E}(e^{tX_1}))\}=\sup_{t\in\mathbb{R}}\{tz-\log(\cosh t)\}$$

A simple computation shows that

$$I(z)=\begin{cases}\frac{1-z}{2}\log(1-z)+\frac{1+z}{2}\log(1+z) & \text{for}|z|\leq 1\\ \infty & \text{for}|z|>1\end{cases}$$

From Varadhan lemma it follows that

$$\lim_{N\to\infty}\frac{1}{N}\log\int_{\mathbb{R}}\exp\left[N(\frac{\beta}{2}z^2+\beta hz)\right]Q_N(dz)=\sup_{z\in[-1,1]}\left\{\frac{\beta}{2}z^2+\beta hz-I(z)\right\}$$

Maximum points of

$$g(z)=\frac{\beta}{2}z^2+\beta hz-I(z)$$

satisfies the equation $\frac{\partial g}{\partial z}=0$, that is

$$z=\mathrm{tgh}(\beta(z+h))$$

The solutions $z(\beta,h)$ of the previous equation can be studied graphically. The nature of the solutions depends on whether $\beta\leq 1$ or $\beta>1$.

- for $\beta\leq 1$ and any real h there is a unique solution $z(\beta,h)$. As $h\to 0$ this solution tends to 0.
- for $\beta>1$ and any real $h\neq 0$ there is a unique solution $z(\beta,h)$ that has the same sign as h; for $\beta>1$ and $h=0$ there are two non zero solutions $z_+(\beta)$ and $z_-(\beta)=-z_+(\beta)$. As $h\to 0^+$, $z(\beta,h)\to z_+(\beta)>0$ and as $h\to 0^-$, $z(\beta,h)\to z_-(\beta)<0$

One can immediately checked that the magnetization $m(\beta,h)$ for the Curie-Weiss model equals $z(\beta,h)$ for $\beta>0,h\neq 0$ and for $\beta<1,h=0$. However the following holds for each choice of sign

$$\lim_{h\to 0^\pm}m(\beta,h)=\begin{cases}z(\beta,0)=0 & \text{for}\beta\leq 1\\ z_\pm(\beta)\neq 0 & \text{for}\beta>1\end{cases}$$

Thus there exist a phase transition in the Curie-Weiss model which is associated to a spontaneous magnetization. One can also study the susceptibility and find that

$$\chi(\beta,h)=\frac{\beta(1-m^2(\beta,h))}{1-\beta(1-m^2(\beta,h))}$$

Notice that this becomes infinite at the critical point $h=0$, $\beta=1$ since $m(1,0)=0$.

3 Ising model on $\mathbb{Z}$

In this last section we solve the Ising model in one dimension. As already anticipated we will find that no phase transition occurs at any positive temperature. The Hamiltonian is given by

$$H_N(\sigma) = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i \tag{3.1}$$

We impose periodic boundary condition (this is not a restriction since the thermodynamic limit of the free energy is not sensitive to the kind of boundary conditions)

$$\sigma_{N+1} = \sigma_1 \tag{3.2}$$

making the topology of a chain that of a circle. The partition function

$$Z_N(\beta, h) = \sum_{\sigma \in \Omega_N} e^{-\beta H(\sigma)} \frac{1}{2^N} \tag{3.3}$$

can be expressed in terms of the product of a *transfer matrix*. Let us write

$$Z_N(\beta, h) = \sum_{\sigma_1 = \pm 1} \dots \sum_{\sigma_N = \pm 1} \exp(\beta \sum_{i=1}^{N} [J\sigma_i \sigma_{i+1} + \frac{1}{2} h(\sigma_i + \sigma_{i_+1})]) \frac{1}{2^N} \tag{3.4}$$

which is equivalent to (3.3) by virtue of (3.2). Let a 2×2 matrix $B(\sigma_i, \sigma_{i+1})$ be defined for $(\sigma_i, \sigma_{i+1}) \in \{-1, +1\}^2$ by the function

$$B(\sigma, \sigma_{i+1}) = \frac{1}{2} \exp[\beta J \sigma_i \sigma_{i+1} + \beta h(\sigma_i + \sigma_{i+1})]$$

With this definition we may rewrite (3.4) in the form

$$\begin{aligned} Z_N(\beta, h) &= \sum_{\sigma_1 = \pm 1} \dots \sum_{\sigma_N = \pm 1} B(\sigma_1, \sigma_2) B(\sigma_2, \sigma_3) \dots B(\sigma_N, \sigma_{N+1}) \\ &= \sum_{\sigma_1 = \pm 1} B^N(\sigma_1, \sigma_1) \\ &= \text{Trace}(B^N) = \lambda_+^N + \lambda_-^N \end{aligned} \tag{3.5}$$

where λ_+ and λ_- are the two eigenvalue of B, with $\lambda_+ > \lambda_-$. By an easy calculation we find that the two eigenvalues are

$$\lambda_{\pm} = \frac{1}{2} e^{\beta J} \left[\cosh(\beta h) \pm \sqrt{\sinh^2(\beta h) + e^{-4\beta J}} \right]$$

Thus $\lambda_+ > \lambda_-$ for all h and as $N \to \infty$ only the larger eigenvalue is relevant becuase

$$\frac{1}{N} \log Z_N(\beta, h) = \log \lambda_+ + \frac{1}{N} \log(1 + (\frac{\lambda_-}{\lambda_+})^N) \to \log \lambda_+ \qquad \text{as} \quad N \to \infty$$

Thus the thermodynamic limit of the pressure is given by

$$p(\beta, h) = -\log 2 + \beta J + \log \left[\cosh(\beta h) \pm \sqrt{\sinh^2(\beta h) + e^{-4\beta J}} \right] \tag{3.6}$$

From this we deduce that the magnetization is

$$m(\beta, h) = \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}$$

For all $0 \leq \beta < \infty$ we see that the spontaneous magnetization $\lim_{h \to 0+} m(\beta, h) = 0$, thus implying the absence of a phase transition.

References

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