Dr. Cristian Giardinà

Stochastic Processes 2WB08 problem set 3

Martingale convergence theorem.

- Problem 6.22, 6.23 a), 6.24.
- Problem 1: Let $(M_n)_{n\in\mathbb{N}_0}$ be a martingale, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function with the property that $E[|\varphi(M_n)|] < \infty$ for all $n \geq 0$. Then, $(\varphi(M_n))_{n\in\mathbb{N}_0}$ is a submartingale.
- Problem 2: Define a sequence $(a_i)_{i\in\mathbb{N}}$ by

  $$a_1 = 2 \qquad \text{and} \qquad a_n = 4\sum_{i=1}^{n-1} a_i \quad \text{for } n \geq 2.$$

  Let X_i, $i \geq 1$, be independent random variables with

  $$P(X_i = a_i) = \frac{1}{2i^2}, \quad P(X_i = 0) = 1 - \frac{1}{i^2}, \quad P(X_i = -a_i) = \frac{1}{2i^2}.$$

  Prove the following:

  1. $(M_n = \sum_{i=1}^n X_i)_{n\in\mathbb{N}_0}$ is a martingale.
  2. $\lim_{n\to\infty} M_n$ exists almost surely (use a Borel-Cantelli argument).
  3. There exists no C such that $E[|M_n|] \leq C$ for all n.

- Problem 3: Let X_i, $i \geq 1$, be independent random variables with

  $$P(X_i = 1) = \frac{1}{2i}, \quad P(X_i = 0) = 1 - \frac{1}{i}, \quad P(X_i = -1) = \frac{1}{2i}.$$

  Set $M_1 = X_1$ and

  $$M_n = \begin{cases} X_n & \text{if } M_{n-1} = 0, \\ nM_{n-1}|X_n| & \text{if } M_{n-1} \neq 0 \end{cases}$$

  for $n \geq 2$.

  1. Show that $(M_n)_{n\in\mathbb{N}_0}$ is a martingale.
  2. Show that $M_n \to 0$ in probability as $n \to \infty$.
  3. Show that M_n does not converge to 0 almost surely.