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Toegepaste kansrekening 2WS15
problem set 3

- Read Lecture 1 (download at http://www.win.tue.nl/ cgiardin).

  Let $(X_i)_{i\geq 1} \in \Omega = \{x_1, x_2, \ldots, x_r\}$ be sequence of discrete i.i.d. random variables with law $\rho = (\rho_1, \ldots, \rho_r)$. Let $\mathcal{M}_1 = \{\nu = (\nu_1, \ldots, \nu_r) : 0 \leq \nu_i \leq 1, \sum_{i=1}^r \nu_i = 1\}$ be the set of probability measures on Ω.

- Problem 1. Consider the relative entropy

$$I_\rho(\nu) = \sum_{i=1}^{r} \nu_i \log\left(\frac{\nu_i}{\rho_i}\right)$$

  Show that
  i) $I_\rho(\nu)$ is finite, continuous and strictly convex on $\mathcal{M}_1$.
  ii) $I_\rho(\nu) \geq 0$ with equality if and only if $\nu = \rho$.

- Problem 2. Consider the entropy

$$I(\nu) = -\sum_{i=1}^{r} \nu_i \log \nu_i$$

  i) Show that $I(\nu)$ attains its maximum on $\mathcal{M}_1$ at ν^*, the uniform measure on Ω. What is the value of $I(\nu^*)$?
  ii) Which measures ν do have $I(\nu) = 0$?

- Problem 3. Let $\Gamma(z) = \{\nu \in \mathcal{M}_1 : \sum_{i=1}^r x_i \nu_i = z\}$. Show that $I(\nu)$ attains its maximum over $\Gamma(z)$ at $\bar{\nu}$ given by

$$\bar{\nu}_i = \frac{\exp\{-\beta x_i\}}{\sum_{j=1}^r \exp\{-\beta x_j\}}$$

  where β is determined by $\sum_{i=1}^r x_i \bar{\nu}_i = z$.

- Problem 4. Using the definitions of free energy $F(\beta)$, internal energy $U(\beta)$, thermodynamic entropy $S(\beta)$ which are given in Lecture 1 show that:
  i) $F(\beta) = U(\beta) - \frac{1}{\beta} S(\beta)$
  ii) $U(\beta) = \frac{\partial}{\partial \beta}(\beta F(\beta))$
  $iii) S(\beta) = \beta^2 \frac{\partial}{\partial \beta}(F(\beta))$.

- Problem 5. Compute the free energy and the internal energy for the perfect gas with Hamiltonian $H_n(p) = \sum_{i=1}^n \frac{p_i^2}{2}$, where $p_i \in \mathbb{R}$ represents particles velocity.

Hints and answers

- Problem 1.
  i) Consider the convex function $x \mapsto h(x) = x \log x$
  ii) Use Jensen inequality: if $h(x)$ is a convex function then $\mathbb{E}(h(X)) \geq h(\mathbb{E}(X))$

- Problem 2.
  i) Write $\sum_{i=1}^{r} \nu_i \log \nu_i = n \sum_{i=1}^{r} \frac{1}{n} \nu_i \log \nu_i$ and use again convexity of $x \mapsto h(x) = x \log x$. Alternatively solve the maximization problem by introducing a Lagrange multiplier to fix the constraint $\sum_{i=1}^{r} \nu_i = 1$. The result is $I(\nu^*) = \log r$.
  ii) These are the r "non-random" measures of the form $\nu = (0, \ldots, 0, 1, 0 \ldots, 0)$.

- Problem 3.
  Solve the maximization problem by introducing two Lagrange multiplier to fix the constraints $\sum_{i=1}^{r} \nu_i = 1$ and $\sum_{i=1}^{r} x_i \nu_i = z$.

- Problem 5.
  Gaussian integration immediately gives $F(\beta) = -\frac{N}{2\beta} \log \left(\frac{2\pi}{\beta} \right)$ and $U(\beta) = \frac{N}{2\beta}$.