Dr. Cristian Giardinà

Stochastic Processes 2WB08 problem set 4

Martingale stopping theorem.

- Problem 6.10, 6.11, 6.13.
- **Problem 1 (Polya urn)**: An urn contains one red and one blue ball. Repeatedly, a ball is drawn at random, its color is noted, and the ball is returned to the urn together with an additional ball of the same color. Let T be the number of balls drawn until the first blue ball appear. Show that
$$E\left[\frac{1}{T+2}\right] = \frac{1}{4}.$$
- **Problem 2 (Polya urn with random increments)**: Let ξ_i, $i \geq 1$, be a sequence of random variables taking values in $\mathbb{N}_0$. An urn contains one red and one blue ball. Repeatedly, a ball is drawn at random and its color is noted; we assume that the distribution of this color depends only on the current contents of the urn and not on any further information concerning the ξ_i. In the nth drawing, the ball is returned to the urn together with ξ_n additional ball of the same color. Denote by R_n and B_n the number of red and blue balls in the urn after n drawings.

  a) Show that
$$Y_n = \frac{R_n}{R_n + B_n}$$
  defines a martingale.

  b) Let T be the number of balls drawn until the first blue ball appear. Show that
$$E\left[\frac{1+\xi_T}{2+\sum_{i=1}^{T}\xi_i}\right] = \frac{1}{2}.$$
- **Problem 3**: Let ξ_i, $i \geq 1$, be i.i.d. with $P(\xi_i = 1) = p$ and $P(\xi_i = -1) = 1 - p =: q$, where $p < 1/2$. Let $S_n := S_0 + \sum_{i=1}^{n} \xi_i$ and $V_0 := \min\{n \geq 0 : S_n = 0\}$. Show that for all $x \in \mathbb{N}_0$ the following holds:
$$E_x[V_0] = \frac{x}{1-2p}.$$
  Here E_x denotes the expectation with respect to the random walk starting at x at time 0.
- **Problem 4**: Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain with state space $\mathbb{N}_0$ and transition probabilities
$$\begin{aligned} p(x, x+1) =& p_x \quad \text{for } x \geq 0, \\ p(x, x-1) =& q_x \quad \text{for } x \geq 1, \\ p(x, x) =& r_x \quad \text{for } x \geq 0 \end{aligned}$$
  und $p(x, y) = 0$ for all other pairs (x, y); here $p_x, q_x, r_x \geq 0$ with $p_x + q_x + r_x = 1$. For $y, z \in \mathbb{N}_0$ let $V_y := \min\{n \geq 0 : X_n = y\}$ and $\phi(z) = \sum_{y=1}^{z} \prod_{x=1}^{y-1} q_x/p_x$. Let $a, b \in \mathbb{N}_0$ with $a < b$. Show that for all $x \in]a, b[\cap \mathbb{N}_0$ the following hold:
$$P_x(V_b < V_a) = \frac{\phi(x) - \phi(a)}{\phi(b) - \phi(a)}.$$