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Toegepaste kansrekening 2WS15
problem set 4

- Problem 1.

  Assume that the law of X_n satisfies the LDP with rate n and rate function

$$I(x) = \begin{cases} ex^{ex} - 1 & , \text{ for } x > 0 \\ e - 1 & , \text{ for } x = 0 \\ \infty & , \text{ for } x < 0 \end{cases}$$

  Show that there is a number μ such that X_n converges to μ in probability. i.e., $\lim_{n\to\infty} \mathbb{P}(|X_n - \mu| < \epsilon) = 1$ for all $\epsilon > 0$. Also, determine μ.

- Problem 2.

  Do exercise III.9 from the book.

- Problem 3.

  Assume that $\{X_i : i \in \mathbb{N}\}$ are i.i.d normal random variables with mean zero and variance 1, and let $S_n := X_1 + \cdots + X_n$. Compute

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} \left(\left(1 + \frac{S_n}{n} 1_{|S_n|<n} \right)^n \right)$$

  Take as true that the law of S_n/n satisfies the LDP with rate n and rate function $I(x) = x^2/2$.

- Problem 4.

  Assume (X_n, Y_n) is a random sequence in $\mathbb{R}^2$ whose law satisfies the LDP with rate n and rate function $I(x, y) = x^2 + (x - y)^2$.

  Prove that the law of the sequence $X_n Y_n$ satisfies the LDP with rate n, and compute the rate function.

Hints and answers

- Problem 1.
  We should show that for some μ it holds

$$\lim_{n\to\infty} \mathbb{P}(X_n \geq \mu + \epsilon) = \lim_{n\to\infty} \mathbb{P}(X_n \leq \mu - \epsilon) = 0$$

  We use large deviations to bound the probabilities.

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathbb{P}(X_n \geq \mu + \epsilon) \leq - \inf_{x\geq\mu+\epsilon} I(x)$$

  and similarly for the other probability. If the last inf is say 3, then we will have that for any $\delta > 0$ (we pick $\delta = 0.5$) there is n_0 so that for $n \geq n_0$ we have that the probability is less than $e^{(-3+\delta)n}$, is particular it goes to zero.

  We need a μ with $\inf_{x\geq\mu+\epsilon} I(x) > 0, \inf_{x\leq\mu-\epsilon} I(x) > 0$. Study the rate function, show that it takes its minimum value (which is zero) at $\mu = e^{-1}$. Left of μ, I is strictly decreasing, to the right it is strictly increasing.

- Problem 2.
  Solution on F.Den Hollander book.

- Problem 3.
  The limit equals

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(e^{n \log\left(1+\frac{S_n}{n} 1_{|S_n|<n}\right)}\right) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(e^{nF(X_n)}\right)$$

  where $X_n = S_n/n$ satisfies the LPD with rate function I, and $F(x) := \log(1 + x1_{|x|<1})$ it is bounded above (by $\log 2$). Apply Varadhan lemma. Find $\sup_{|x|<1}(F(x) - I(x))$. In fact it is the same as $\sup_{x>-1}(\log(1 + x) - I(x)) = F(x_0) - I(x_0)$ with $x_0 = (-1 + \sqrt{5})/2 \in (0, 1)$.

- Problem 4.
  We apply the contraction principle for the continuous map $T : \mathbb{R}^2 \to \mathbb{R}$ with $T(x, y) = xy$. So the rate is

$$J(z) = \inf_{x,y:xy=z} I(x, y) = \inf_{x\in\mathbb{R}} I(x, \frac{z}{x}) = ... = 2(\sqrt{2}|z| - z)$$