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Stochastic Processes 2WB08 – problem set 5

Brownian Motion I

- Problems 8.1abc, 8.2, 8.3, 8.4, 8.5 from the book.

- **Problem 1**: Let $U_1, U_2, \ldots$ be independent random variables with the uniform distribution on [0,1]. Let $I_j(x)$ be the indicator of the event $\{U_j < x\}$ and define

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} I_j(x) \qquad\qquad 0 < x < 1$$

  The function $F_n(x)$ is called the empirical distribution function of the U_j.

  1. Find the mean and the variance of $F_n(x)$ and prove that $\sqrt{n}(F_n(x) - x)$ converges in distribution as $n \to \infty$ to $Y(x)$ which is normally distributed with mean zero and variance $x(1-x)$
  2. What is the multivariate limit distribution of a collection of random variables of the form $\{\sqrt{n}(F_n(x_i) - x_i) : 1 \leq \imath \leq k\}$ where $0 \leq x_1 < x_2 < \ldots x_k < 1$?
  3. Compare to Brownian bridge

- **Problem 2**: Prove the following useful identities for gaussian random variables:

  1. For a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$

$$\mathbb{E}\left(e^{aX}\right) = e^{a\mu + \frac{\sigma^2 a^2}{2}}$$

  2. For a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ and a function $f : \mathbb{R} \to \mathbb{R}$ deduce the following "integration by parts" formula

$$\mathbb{E}(Xf(X)) = \mu\mathbb{E}(f(X)) + \sigma^2\mathbb{E}\left(\frac{\partial f}{\partial x}(X)\right)$$

  3. Generalization to multivariate gaussian. For a family of normal random variables $X = (X_i)_{i=1,\ldots n}$ having mean $\mathbb{E}(X_i) = \mu_i$ and covariance matrix $\mathbb{E}((X_i - \mu_i)(X_j - \mu_j)) = c_{i,j}$ prove that

$$\mathbb{E}\left(e^{\sum_{i=1}^{n} a_i X_i}\right) = e^{\sum_{i=1}^{n} a_i\mu_i + \frac{1}{2}\sum_{i,j=1}^{n} c_{i,j} a_i a_j}$$

     and for a function $f : \mathbb{R}^n \to \mathbb{R}$ prove the following integration by parts formula (or 'Wick law')

$$\mathbb{E}(X_i f(X)) = \mu_i\mathbb{E}(f(X)) + \sum_{j=1}^{n} c_{i,j}\mathbb{E}\left(\frac{\partial f}{\partial x_j}(X)\right)$$