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# Toegepaste kansrekening 2WS15
# Problem set 5

- Problem 1.

  Assume that the sequence of random variables $(X_n)_{n\geq 1}$ satisfies for every $z > 0$

  $$\mathbb{E}(z^{X_n}) = (n+2)e^{na(z)} + (n^3 - 1)e^{nb(z)}$$

  where $a(z) = (-3 + \sqrt{5+4z})/2$ and $b(z) = -2 + \sqrt{z}$.

  1. Show that the laws $(P_n)_{n\geq 1}$ of $(X_n/n)_{n\geq 1}$ satisfy the LDP, and compute the rate function.
  2. Show that $X_n/n \to 1/3$ in probability.

- Problem 2.

  A function $f : I \to (-\infty, \infty]$, with domain an interval $I \subset \mathbb{R}$, is called convex if

  $$f(ax + (1-a)y) \leq af(x) + (1-a)f(y)$$

  for all $x, y \in I$ and $a \in [0,1]$. Assume that $\{f_t(x) : t \in \mathbb{R}\}$ is a family of functions with domain $I$ and values in $(-\infty, \infty]$, and let $f(x) := \sup_{t\in\mathbb{R}} f_t(x)$

  1. If all $f_t$ are convex, then $f$ is convex.
  2. Conclude that the Legendre transform of a function $f : \mathbb{R} \to \mathbb{R}$ is convex.
  3. If all $f_t$ are lower semicontinuous, then $f$ is lower semicontinuous.

- Problem 3. Let $\Lambda : \mathbb{R} \to [-\infty, \infty]$ be a function and $\Lambda^* : \mathbb{R} \to [-\infty, \infty]$ its Legendre transform.

  1. If $0 \in D^o_\Lambda$, then $\Lambda^*$ has compact level sets.
  2. If $D_\lambda = \mathbb{R}$ (i.e. $\Lambda$ is always less than $\infty$), then $\lim_{|x|\to\infty} \Lambda^*(x)/|x| = \infty$.

**Hints and answers**

- Problem 1.
  Use the Gartner-Ellis theorem. Explain why all the assumptions of the theorem are satisfied. The rate function equals

$$x \log\left(2x(x+\sqrt{x^2+5/4})\right)+\frac{3}{2}-(x+\sqrt{x^2+5/4})$$

  For the seconds statement, study the rate function. Recall problem 1 of the previous homework.

- Problem 2.

  Easy.

- Problem 3.

  First part.

  Pick $t_1 < 0 < t_2$ with $\Lambda(t_1), \Lambda(t_1) < \infty$. Then for every $x$, $\Lambda^*(x) \geq xt_1 - \Lambda(t_1)$ and $\Lambda^*(x) \geq xt_2 - \Lambda(t_2)$. Look at lemma V.4 (ii) for the remaining details.

  Second part.

  For $x < 0$, we have $\Lambda^*(x)/x \leq t_1 - \Lambda(t_1)/$. First take $x \to -\infty$ to get

$$\overline{\lim_{x\to-\infty}} \frac{\Lambda^*(x)}{x} \leq t_1.$$

  Next?