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Stochastic Processes 2WB08

Brownian Motion II

- Problems 8.7, 8.8, 8.9, 8.10, 8.11, 8.12 from the book.

- **Problem 1**: Let $(B_t)_{t\geq 0}$ be a Brownian motion (starting at 0). Let a, b be real numbers with $a < 0$ and $b > 0$. For any real number x, denote by τ_x the first time the Brownian motion hits x:
$$\tau_x := \inf\{t \geq 0 : B_t = x\}.$$
Show that
$$P(\tau_a < \tau_b) = \frac{b}{b-a}.$$
Hint: Use the following *stopping theorem for continuous martingales*: Let $(M_t)_{t\geq 0}$ be a martingale such that $t \mapsto M_t$ is continuous. Let T be a stopping time with $P(T < \infty) = 1$. Assume there is a constant C such that $|M_{T\wedge t}| \leq C$ holds for all times $t \geq 0$. (Of course, C is assumed to be independent of t.) Then, $E[M_T] = E[M_0]$ holds.

- **Problem 2**: We use the same notation as in Problem 1. Denote by T the first time, the Brownian motion exits the interval (a, b):
$$T := \inf\{t \geq 0 : B_t \notin (a, b)\}.$$
Show that $E[T] = -ab$.

- **Problem 3**: We use the same notation as in Problems 1 and 2. Prove that τ_a, $a \geq 0$, has stationary independent increments. In other words:
  - if $a < b$, then $\tau_b - \tau_a$ has the same distribution as τ_{b-a}.
  - if $a_0 = 0 < a_1 < \cdots < a_n$, then $\tau_{a_{i+1}} - \tau_{a_i}$, $0 \leq i \leq n-1$, are independent.

  Hint: Use the *strong Markov property of Brownian motion*: If T is a stopping time, then $(B_{T+t} - B_T)_{t\geq 0}$ is a Brownian motion, independent of $(B_t)_{t\leq T}$.

- **Problem 4**: In the following, $(B_t)_{t\geq 0}$ is a Brownian motion.
  (a) For $t \geq 0$, set $X_t = |B_t|$. The process $(X_t)_{t\geq 0}$ is called a *Brownian motion reflected at the origin*. Show that
$$E[X_t] = \sqrt{2t/\pi}, \quad \mathrm{Var}(X_t) = \left(1 - \frac{2}{\pi}\right) t.$$
  (b) For $t \geq 0$, set $Y_t = e^{B_t}$. The process $(Y_t)_{t\geq 0}$ is called a *geometric Brownian motion*. Show that
$$E[Y_t] = e^{t/2}, \quad \mathrm{Var}(Y_t) = e^{2t} - e^t.$$
  (a) For $t \geq 0$, set $Z_t = \int_0^t B_s\, ds$. The process $(Z_t)_{t\geq 0}$ is called an *integrated Brownian motion*. Show that for $s \leq t$,
$$E[Z_t] = 0, \quad \mathrm{Cov}(Z_s, Z_t) = s^2\left(\frac{t}{2} - \frac{s}{6}\right).$$