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Toegepaste kansrekening 2WS15
Problem set 6

Consider the Curie-Weiss model with Hamiltonian

$$H_N(\sigma) = -\frac{1}{N}\sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_1$$

- Problem 0.

  Study Lecture 2 from the web-site (It will be difficult to do exercises below without reading the notes.)

- Problem 1.

  Verify with all details the large deviation solution of the model. Specifically:

  1. Obtain the expression for the pressure

  $$\begin{aligned} p(\beta, h) &= \lim_{\mathbb{N}\to\infty} \frac{1}{N} \log \left(\sum_{\sigma\in\{-1,+1\}^N} e^{-\beta H_N(\sigma)} \right) \\ &= \sup_{z\in[-1,1]} \left\{ \frac{1}{2}\beta z^2 + \beta z h - \frac{1-z}{2}\log(1-z) - \frac{1+z}{2}\log(1+z) \right\} \end{aligned}$$

  2. Obtain the equation for the magnetization

  $$m(\beta, h) = \lim_{\mathbb{N}\to\infty} \frac{\sum_\sigma \left(\frac{1}{N}\sum_{i=1}^N \sigma_i\right) e^{-\beta H_N(\sigma)}}{\sum_\sigma e^{-\beta H_N(\sigma)}}$$

  3. Show graphically the existence of a non-zero spontaneous magnetization $m(\beta) = \lim_{h\to 0^+} m(\beta, h)$ for $\beta > 1$.

- Problem 2.

  There is another solution for the model without using large deviations.

  1. By using the identity

  $$\exp(\frac{y^2}{2}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(ty - \frac{t^2}{2}) dt$$

  show that

  $$p(\beta, h) = \sup_t \left\{ \log\cosh t - \frac{1}{2\beta}(t - \beta h)^2 \right\} .$$

2. What is the equation for the critical point of the function inside the sup in the above item?
3. Prove that if f and g are convex functions on $\mathbb{R}$

$$\sup_{x \in dom(g)} \{f(x) - g(x)\} = \sup_{y \in dom(f^*)} \{g^*(y) - f^*(y)\}$$

where f^* denotes the Legendre transform of f.
4. Using the previous item show that

$$\sup_z \left\{ \frac{1}{2}\beta z^2 + \beta z h - \frac{1-z}{2}\log(1-z) - \frac{1+z}{2}\log(1+z) \right\} = \sup_t \left\{ \log\cosh t - \frac{1}{2\beta}(t - \beta h)^2 \right\}$$

thus establishing that the two solutions of the CW model are equivalent!

- Problem 3.

1. Prove that

$$\chi(\beta, h) = \frac{\beta(1 - m^2(\beta, h))}{1 - \beta(1 - m^2(\beta, h))}$$

where $m(\beta, h)$ is the magnetization.
2. What happens for $h = 0$ and $\beta = 1$?

Hints

- Problem 1.

  Look at the note of Lecture 2.

- Problem 2.

  Item 3.: from the definition of Legendre transform it immediately follows that

$$f^*(y) + f(x) \geq xy \tag{1}$$

  Use (1) and the "trick" $f(x) - g(x) = -(xy - f(x)) + xy - g(x)$ to show that

$$\sup_{x \in dom(g)} \{f(x) - g(x)\} \geq \sup_{y \in dom(f^*)} \{g^*(y) - f^*(y)\}$$

  Use (1) and the fact that $(f^*)^*(x) = f(x)$ and the "trick" $g^*(y) - f^*(y) = xy - f^*(y) - (xy - g^*(y))$ to show that

$$\sup_{y \in dom(f^*)} \{g^*(y) - f^*(y)\} \geq \sup_{x \in dom(g)} \{f(x) - g(x)\}$$

  Combine the two bounds together.

- Problem 3. Take derivative w.r.t to h of the implicit equation for the magnetization (Prob. 1 item 2.)