SPIN-GLASS STOCHASTIC STABILITY: A RIGOROUS PROOF.

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Abstract

We prove the property of stochastic stability previously introduced as a consequence of the (unproved) continuity hypothesis in the temperature of the spin-glass quenched state. We show that stochastic stability holds in β-average for both the Sherrington-Kirkpatrick model in terms of the square of the overlap function and for the Edwards-Anderson model in terms of the bond overlap. We show that the volume rate at which the property is reached in the thermodynamic limit is V^{-1}. As a byproduct we show that the stochastic stability identities coincide with those obtained with a different method by Ghirlanda and Guerra when applyed to the thermal fluctuations only.

1 Introduction

In a previous paper by Aizenman and Contucci [AC] the property of stochastic stability was introduced as the consequence of a continuity (in term of the inverse temperature β) hypothesis of the quenched state for the Sherrington-Kirkpatrick [SK] model. Stochastic stability says that a suitable class of perturbations of the spin glass Hamiltonian produces very small changes in the quenched equilibrium state and that such a change vanishes in the thermodynamic limit. This property has interesting consequences for the spin glass models: in terms of the overlap distribution it implies that the quenched measure is *replica-equivalent* [MPV, P] a property originally introduced within the replica symmetry breaking Parisi ansatz. The same property is also used in [FMPP1, FMPP2] to build a bridge between equilibrium and off-equilibrium properties in a spin-glass model being these last the only ones physically accessible to experimental investigation. More recently all and only the constraints that stochastic stability implies for the overlap moments have been completely classified [C, BCK].

In this paper we give a rigorous proof of stochastic stability property in *β-average*. This result is achieved in an elementary way by use of the sum law for independent Gaussian variables and works in full generality for both mean-field and finite dimensional spin glass models. We also derive the explicit form of the stochastic stability identities which first appeared in [AC] and we prove, using integration by parts in the spirit of [CDGG], that they coincide with a subset of the Ghirlanda-Guerra identities [G, GG], namely the part related to the thermal fluctuation bound (see also [T] for a nice set of rigorous results derived from those identities).

The proof also provides the rate at which stochastic stability in β-average is reached with the thermodynamic limit which turns out to be V^{-1}. The paper is organized with Sec. 2 containing a list of the definitions and the statement of the two main theorems. Their proof is built in Sec. 3 while Sec. 4 shows how to apply the results to both the mean field models, which we illustrate for the Sherrington-Kirkpatrick model [SK], and for the finite dimensional cases with the Edwards-Anderson model [EA]. Sec. 5 collects some comments.

2 Definitions and Results

We consider a disordered model of Ising configurations $\sigma_n = \pm 1$, $n \in \Lambda \subset \mathbb{Z}^d$ for some d-parallelepiped Λ of volume $|\Lambda|$. We denote Σ_Λ the set of all $\sigma = \{\sigma_n\}_{n\in\Lambda}$, and $|\Sigma_\Lambda| = 2^{|\Lambda|}$. In the sequel the following definitions will be used.

1. Hamiltonian. For every $\Lambda \subset \mathbb{Z}^d$ let $\{H_\Lambda(\sigma)\}_{\sigma\in\Sigma_N}$ be a family of $2^{|\Lambda|}$ *translation invariant (in distribution) centered Gaussian* random variables of *volume-size* covariance matrix

$$\mathrm{Av}\left(H_\Lambda(\sigma)H_\Lambda(\tau)\right) \ = \ |\Lambda|\, \mathcal{Q}_\Lambda(\sigma,\tau)\,, \tag{2.1}$$

and

$$\mathcal{Q}_\Lambda(\sigma,\sigma) \ = \ 1\,. \tag{2.2}$$

By the Schwarz inequality $|\mathcal{Q}_\Lambda(\sigma,\tau)| \leq 1$ for all σ and τ.

2. Random partition function

$$\mathcal{Z}(\beta) \ := \sum_{\sigma\in\,\Sigma_\Lambda} e^{-\beta H_\Lambda(\sigma)}\ . \tag{2.3}$$

3. Random free energy $\mathcal{F}(\beta)$

$$-\beta\mathcal{F}(\beta) \ := \ \mathcal{A}(\beta) \ := \ \ln \mathcal{Z}(\beta)\ . \tag{2.4}$$

4. Quenched free energy $F(\beta)$

$$-\beta F(\beta) \ := \ A(\beta) \ := \ \mathrm{Av}\left(\mathcal{A}(\beta)\right)\ . \tag{2.5}$$

5. R-product random Gibbs-Boltzmann state

$$\Omega(-) \ := \sum_{\sigma^{(1)},\ldots,\sigma^{(R)}} (-)\, \frac{e^{-\beta[H_\Lambda(\sigma^{(1)})+\cdots+H_\Lambda(\sigma^{(R)})]}}{[\mathcal{Z}(\beta)]^R}\ . \tag{2.6}$$

6. Quenched equilibrium state

$$< - > := \mathrm{Av}\left(\Omega(-)\right)\ . \tag{2.7}$$

7. Observables. For any smooth bounded function $G(\mathcal{Q}_\Lambda)$ (without loss of generality we consider $|G| \leq 1$) of the covariance matrix entries we introduce the random (with respect to $< - >$) $R \times R$ matrix $Q = \{q_{k,l}\}$ by the formula

$$< G(Q) > := \mathrm{Av}\,(\Omega(G(\mathcal{Q}_\Lambda))) \ . \tag{2.8}$$

E.g.: $G(\mathcal{Q}_\Lambda) = \mathcal{Q}_\Lambda(\sigma^{(1)}, \sigma^{(2)})\mathcal{Q}_\Lambda(\sigma^{(2)}, \sigma^{(3)})$

$$< q_{1,2}q_{2,3} > = \mathrm{Av}\left(\frac{\sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} \mathcal{Q}_\Lambda(\sigma^{(1)}, \sigma^{(2)})\mathcal{Q}_\Lambda(\sigma^{(2)}, \sigma^{(3)})\, e^{-\beta[\sum_{i=1}^{3} H_\Lambda(\ s^{(i)})]}}{[\mathcal{Z}(\beta)]^3}\right) \tag{2.9}$$

8. Deformed quenched state. For every $\Lambda \subset \mathbb{Z}^d$ let the $\{K_\Lambda(\sigma)\}_{\sigma\in\Sigma_N}$ be a *translation invariant centered Gaussian* random family of *size one* covariance matrix

$$\mathrm{Av}\,(K_\Lambda(\sigma)K_\Lambda(\tau)) \ = \ \mathcal{Q}_\Lambda(\sigma, \tau)\,, \tag{2.10}$$

where the families H and K are mutually independent with respect to the joint Gaussian distribution, i.e.

$$\mathrm{Av}\,(H_\Lambda(\sigma)K_\Lambda(\tau)) \ = \ 0\ . \tag{2.11}$$

We consider

$$\mathcal{Z}_\lambda(\beta) \ := \ \sum_{\sigma\in\,\Sigma_\Lambda} e^{-\beta H_\Lambda(\sigma)+\sqrt{\lambda}K_\Lambda(\sigma)}\ , \tag{2.12}$$

$$A_\lambda(\beta) \ := \ \mathrm{Av}\,(\ln \mathcal{Z}_\lambda(\beta)) \ , \tag{2.13}$$

$$\Omega_\lambda(-) \ := \ \frac{\Omega((-)\, e^{\sqrt{\lambda}\,[K_\Lambda(\sigma^{(1)})+\dots+K_\Lambda(\sigma^{(R)})]})}{\Omega(e^{\sqrt{\lambda}\,[K_\Lambda(\sigma^{(1)})+\dots+K_\Lambda(\sigma^{(R)})]})}\ , \tag{2.14}$$

and the deformed quenched state

$$< - >_\lambda := \mathrm{Av}\,(\Omega_\lambda(-)) \ . \tag{2.15}$$

9. Stochastic Stability. The quenched measure is said to be *stochastically stable* if for every observable G (see def. 7) the deformed state is stationary in the thermodynamic limit:

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{d}{d\lambda} < G >_\lambda = \; 0 \tag{2.16}$$

It is possible to see (within Theorem 2) that there is a function of the overlap matrix elements: ΔG s.t.

$$< \Delta G >_\lambda := \frac{d}{d\lambda} < G >_\lambda \; . \tag{2.17}$$

A stochastically stable measure fulfills then the property

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} < \Delta G >_\lambda = \; 0 \tag{2.18}$$

for all the observables G.

Our main result state that a spin glass model is stochastically stable β-*almost everywhere* (Theorem 1), characterizes the functions ΔG (Theorem 2) and establish their coincidence with the quantities obtained with the Ghirlanda-Guerra method when applied only to the thermal fluctuations.

Theorem 1 (Stochastic Stability) *The spin-glass quenched state is stochastically stable* in β-average, *i.e. for each interval* $[\beta_1, \beta_2]$ *and each observable* G *(as in def. 7):*

$$\left| \int_{\beta_1^2}^{\beta_2^2} < \Delta G >_\lambda \; d\beta^2 \right| \; \leq \; \frac{2}{|\Lambda|} \; . \tag{2.19}$$

Theorem 2 (Zero average Observables) *The explicit form of the zero average quantities is*

$$2\Delta G \; = \sum_{\substack{k,l=1 \\ k \neq l}}^{R} G \, q_{l,k} - 2RG \sum_{l=1}^{R} q_{l,\, R+1} + R(R+1) G \, q_{R+1,\, R+2} \; , \tag{2.20}$$

which coincide with thermal part of the Ghirlanda-Guerra identities.

3 Proof of the results

Proof of Theorem 1: Since for $\widetilde{H}$ independent from H and K and distributed like H we have, in distribution, that

$$-\beta H_\Lambda + \sqrt{\lambda} K_\Lambda \stackrel{\mathcal{D}}{=} -\sqrt{\beta^2 + \frac{\lambda}{|\Lambda|}}\, \widetilde{H}_\Lambda \tag{3.21}$$

from the def. (8) of the deformed quenched state of the function G, all the expectations $< G >_\lambda$ turn out to be functions of $\beta^2 + \frac{\lambda}{|\Lambda|}$: $< G >_\lambda = g\left(\beta^2 + \frac{\lambda}{|\Lambda|}\right)$. From the composite function derivation rule we deduce (the prime denotes derivative w.r.t. the argument):

$$\frac{d}{d\lambda} < G >_\lambda = \; g'\left(\beta^2 + \frac{\lambda}{|\Lambda|}\right) \cdot \frac{1}{|\Lambda|} \tag{3.22}$$

and

$$\frac{d}{d\beta} < G >_\lambda = \; g'\left(\beta^2 + \frac{\lambda}{|\Lambda|}\right) \cdot 2\beta\;, \tag{3.23}$$

from which we have

$$2\beta \frac{d}{d\lambda} < G >_\lambda = \; \frac{1}{|\Lambda|} \frac{d}{d\beta} < G >_\lambda \;\; . \tag{3.24}$$

Integrating in $d\beta$ and using the foundamental theorem of calculus we obtain

$$\int_{\beta_1^2}^{\beta_2^2} < \Delta G >_\lambda \;\, d\beta^2 = \frac{< G >_\lambda (\beta_2) - < G >_\lambda (\beta_1)}{|\Lambda|} \tag{3.25}$$

Remembering the assumption on boundedness of function G (def. 7) this complete the proof.

Proof of Theorem 2: let $h(\sigma) = |\Lambda|^{-1} H_\Lambda(\sigma)$ be the Hamiltonian per particle. From formula (3.24) and a direct computation of the derivative of $< G >_\lambda$ with respect to the inverse temperature we have

$$-2\beta < \Delta G >_\lambda = \sum_{l=1}^{R} \mathrm{Av}\left(\Omega_\lambda(h(\sigma^{(l)})\, G) - \Omega_\lambda(h(\sigma^{(l)}))\Omega_\lambda(G)\right) \;\; . \tag{3.26}$$

For each replica l ($1 \leq l \leq R$), we evaluate separatly the two terms in the right side of Eq. (3.26) by using the integration by parts (generalized Wick formula) for correlated

Gaussian random variables, $x_1, x_2, \ldots, x_n$

$$\mathrm{Av}\left(x_i\, \psi(x_1, ..., x_n)\right) = \sum_{j=1}^{n} \mathrm{Av}\left(x_i x_j\right)\ \mathrm{Av}\left(\frac{\partial \psi(x_1, ..., x_n)}{\partial x_j}\right) \ . \tag{3.27}$$

It is convenient to denote by $p_\lambda\,(R)$ the Gibbs-Boltzmann weight of R copies of the deformed system

$$p_\lambda\,(R)\ =\ \frac{e^{-\beta\,[\sum_{k=1}^{R} H_\Lambda(\sigma^{(k)})]\,+\,\sqrt{\lambda}\,[\sum_{k=1}^{R} K_\Lambda(\sigma^{(k)})]}}{[\mathcal{Z}_\lambda(\beta)]^R}\ , \tag{3.28}$$

so that we have

$$-\frac{1}{\beta}\frac{dp_\lambda\,(R)}{dH_\Lambda(\tau)}\ =\ p_\lambda\,(R)\left(\sum_{k=1}^{R}\delta_{\sigma^{(k)},\,\tau}\right) - R\,p_\lambda\,(R)\ \frac{e^{-\beta[H_\Lambda(\tau)]}}{[\mathcal{Z}_\lambda(\beta)]}\ . \tag{3.29}$$

We obtain

$$\begin{aligned}
\mathrm{Av}\left(\Omega_\lambda(h(\sigma^{(l)})\,G)\right)\ &=\ \frac{1}{|\Lambda|}\,\mathrm{Av}\left(\sum_{\sigma^{(1)},\ldots,\sigma^{(r)}} G\ H_\Lambda(\sigma^{(l)})\ p_\lambda\,(R)\right) && (3.30)\\
&=\ \mathrm{Av}\left(\sum_{\sigma^{(1)},\ldots,\sigma^{(r)}}\ \sum_{\tau}\ G\ \mathcal{Q}_\Lambda(\sigma^{(l)},\tau)\ \frac{dp_\lambda\,(R)}{dH_\Lambda(\tau)}\right) && (3.31)\\
&=\ -\beta\left[< G >_\lambda + \sum_{\substack{k=1\\k\neq l}}^{R} < G\,q_{l,\,k} >_\lambda - R < G\,q_{l,\,R+1} >_\lambda\right] && (3.32)
\end{aligned}$$

where in (3.31) we made use of the integration by parts formula and (3.32) is obtained by (3.29). Analogously, the other term reads

$$\begin{aligned}
\mathrm{Av}\left(\Omega_\lambda(h(\sigma^{(l)}))\,\Omega_\lambda(G)\right)\ &=\ \frac{1}{|\Lambda|}\,\mathrm{Av}\left(\sum_{\sigma^{(l)}}\ \sum_{\tau^{(1)},\ldots,\tau^{(R)}} G\ H_\Lambda(\sigma^{(l)})\ p_\lambda\,(R+1)\right) && (3.33)\\
&=\ \mathrm{Av}\left(\sum_{\sigma^{(l)}}\ \sum_{\tau^{(1)},\ldots,\tau^{(R)}}\ \sum_{\gamma}\ G\ \mathcal{Q}_\Lambda(\sigma^{(l)},\gamma)\ \frac{dp_\lambda\,(R+1)}{dH_\Lambda(\gamma)}\right) && (3.34)\\
&=\ -\beta\ [< G >_\lambda + R < G\,q_{l\,R+1} >_\lambda - (R+1) < G\,q_{R+1,\,R+2} >_\lambda] && (3.35)
\end{aligned}$$

Inserting the (3.32) and (3.35) in Eq. (3.26) we finally obtain Theorem 2.

Remark: The proof of the Theorems shows that the identities which follow from the stochastic stability property are included in the Ghirlanda-Guerra identities [GG]. Indeed the family of GG identities are obtained from the self-averaging of the internal energy per particle with respect to the full equilibrium quenched measure. This implies, by the use of the Cauchy-Schwartz inequality, the vanishing of the truncated correlation between internal energy per particle and a generic observable G in the thermodynamic limit:

$$< hG > - < h >< G > \to 0 \qquad as \quad |\Lambda| \to \infty \, . \tag{3.36}$$

But clearly the previous fluctuation can be decomposed as a sum of the thermal fluctuation (averaged over the Gaussian disorder) and the fluctuation with respect to the disorder itself, *i.e.*

$$\begin{aligned} < h\, G > - < h >< G > &= \mathrm{Av}\,(\Omega[h\, G]) - \mathrm{Av}\,(\Omega[h])\,\mathrm{Av}\,(\Omega[G]) \\ &= \mathrm{Av}\,(\Omega[h\, G] - \Omega[h]\Omega[G])) \; + \qquad (3.37) \\ &\quad\ \mathrm{Av}\,(\Omega[h]\Omega[G]) - \mathrm{Av}\,(\Omega[h])\,\mathrm{Av}\,(\Omega[G]) \qquad (3.38) \end{aligned}$$

By formula (3.26) we see that the thermal fluctuations (Eq.(3.37)) are those controlled by the stochastic stability.

4 Models

The results proved in the previous sections hold true in complete generality because they are based on the general property of Gaussian variables. Stochastic stability in particular is fulfilled by both mean field models (like the Sherrington-Kirkpatrick, its p-spin generalization, the REM and GREM models etc.) and by the finite dimensional models (like the Edwards-Anderson and Random Field models in general dimension d). The main point to be observed and well stressed is that each one of these models has his own set of observables which describe the quenched equilibrium state, namely the Gaussian covariance matrix of their own Hamiltonians, see Eq. (2.1). To be more specific let illustrate the two main cases of the covariance matrix for the Sherrington-Kirkpatrick model and for the Edwards-Anderson. The SK model of Hamiltonian

$$H_N(\sigma, J) \;=\; -\frac{1}{\sqrt{N}}\sum_{i,j=1}^{N} J_{i,j}\sigma_i\sigma_j \tag{4.39}$$

with $\{J_{ij}\}$ identical independent normal Gaussian variables has a covariance matrix given by the standard overlap function between two configurations:

$$\mathcal{Q}^{(SK)}_{\Lambda}(\sigma,\tau) \;=\; \left[\frac{1}{N}\sum_{i=1}^{N}\sigma_i\tau_i\right]^2 \tag{4.40}$$

The Edwards-Anderson Hamiltonian is

$$H_{\Lambda}(J,\sigma) \;=\; -\sum_{(n,n')\in B(\Lambda)} J_{n,n'}\sigma_n\sigma_{n'}\ , \tag{4.41}$$

where the $J_{n,n'}$ are again independent normal Gaussian variables and the sum runs over all pairs of nearest neighbors sites $n, n' \in \Lambda \subset \mathbb{Z}^d$ with $|n - n'| = 1$. Using the standard identification of the space of nearest neighbors with the d-dimensional *bond*-lattice $b \in \mathbb{B}^d$ with $b = (n, n')$ and denoting $B(\Lambda)$ the d-bond-parallelepiped associated to Λ ($|B| = d|V|$) we introduce, for two spin configurations σ and τ, the notation $\sigma_b = \sigma_n\sigma_{n'}$ and $\tau_b = \tau_n\tau_{n'}$. The covariance matrix turns out to be

$$\mathcal{Q}^{(EA)}_{\Lambda}(\sigma,\tau) := \frac{1}{|B|}\sum_{b\in B} Q_b(\sigma,\tau)\ . \tag{4.42}$$

where the local bond-overlap $Q_b(\sigma,\tau)$ between σ and τ is

$$Q_b(\sigma,\tau) := \sigma_b\tau_b\ . \tag{4.43}$$

The property of stochastic stability for the Edwards-Anderson model in terms of its link-overlap has been originally considered in [C2]. The theorem proved here provides the generalization to the generic observable G.

5 Comments

In this paper we have proved that every Gaussian spin glass model is stochastically stable with respect to a suitable class of perturbations. The consequences of such a stability can be expressed as zero average observables in terms of the proper overlap that each model carries: the covariance of its own Hamiltonian. It is finally worth to mention that the identities that we proved for the Edwards-Anderson model are compatible with both the pictures of triviality and those of non-triviality for the overlap distribution at low temperature; for a discussion the reader may see the replica symmetry Breaking theory in [MPV], the Droplet theory in [FH, BM], the chaotic theory in [NS] and the trivial-non-trivial in [PY, KM]. Nevertheless the stochastic stability identities could suggest a test of triviality for the suitable overlap distribution in the same spirit of [MPRRZ]. We plan to return on these questions in a future work.

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