Answer to Comment on "Ultrametricity in the Edwards-Anderson Model"
http://arxiv.org/abs/0709.0894

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In a recent comment [1] to our Letter [2] T.Jorg and F.Krzakala have investigated the properties of 2-dimensional Edwards-Anderson (EA) model and found, by numerical methods, the interesting result that some ultrametric features hold for the link overlap probability distributions for square lattices of side 16 and 32. Namely,

$$\rho_X(x) = \delta(x)$$
$$\rho_Y(y) = \frac{1}{4}\delta(y) + \frac{3}{2}\theta(y)\int_y^1 P(a)P(a-y)da \quad (1)$$

where X (resp. Y) is the difference between the medium and the smallest (resp. the difference between the largest and the medium) for a triplet of link overlaps sampled with respect to the equilibrium quenched measure. Their results are illustrated in figure 1 of [1]. Since at positive temperature in two dimensions the RSB picture cannot hold they conclude that the results presented in [2] are not sufficient to dismiss the droplet picture in the 3-dimensional EA model.

Our answer can be summarized as follows.
The conclusions obtained in [2] are *mainly* based on the analysis of the scaling properties of the variance for the two random variables X and Y. The statement in favor of ultrametricity that we made is in fact based on the observation (see Fig. 1 and 3 in [2]) that for increasing volumes the variance of the variable X is shrinking to zero with a suitable scaling law while the variance of the variable Y is not. The subsequent analysis of the distribution shape for some finite volume (see Fig. 2 in [2]) was indeed proposed as a further support of the main result. Since the study in [1] is only concerned with a finite volume analysis of the overlap distributions with NO asymptotic analysis it cannot be used to weaken the conclusions obtained in [2]. In order to parallel the approach followed in [2] one should have performed in fact, prior to analyse the probability distribution for some finite volume, the asymptotic behaviour of the variances of the variables X and Y. This can be done indeed with a modest computational effort and gives the result shown in Fig. 1. One immediately sees that both the variances are shrinking to zero and by consequence ultrametricity doesn't hold. This shows that the method developed in [2] is robust.

Still the observation that in $d = 2$ the overlap distribution shape stays, for the volumes considered, close to that predicted by the RSB picture is interesting and deserves a proper explanation.

Here we notice that in the (T, d) plane - dimension vs. temperature - there is a curve which separates in the thermodynamic limit the region with broken symmetry (the upper one) from the paramagnetic one. At $T = 0$ the curve crosses the d axis on the lower critical dimension $d_l = 2.5$ [3] and it grows for positive temperatures $T > 0$. According to the RSB picture [4] the upper region is characterized by a spin glass phase with an ultrametric overlap distribution. However, for a finite volume system, if one is outside the spin glass region but close enough to the critical curve in the (T, d) plane one might still observe some features of an ultrametric overlap distribution.

The point investigated in [1] $(T = 0.2, d = 2)$ is just below the critical curve and not far enough to observe, for the volumes they investigate which in $d = 2$ are quite small, the thermodynamic properties.

To support our claim it is enough to investigate the point $d = 1$ and $T = 0$. Since we are now really away from the critical curve ultrametricity cannot hold anymore. At zero temperature the relevant states in the Gibbs measure are only ground states, which for a frustrated closed chain are kink and anti-kink (we disregard non frustrated disorder samples which are obviously ferromagnetic).

We analyzed the probability distribution for a triplet of *standard* overlap, since in $d = 1$ the link overlap is 1 with probability 1. As one expects the first signal of violation of ultrametricity can be detected studying the behaviour of the random variable $S = sign(q_{1,2}\,q_{2,3}\,q_{3,1})$ (see also [5]). An explicit computation shows that the quenched expectation $< S >= 1/2$, while for an ultrametric topology, as the one predicted by RSB theory, one should have $< S >= 1$. Moreover, for the distribution of $\tilde{X} = \tilde{q}_{med} - \tilde{q}_{min}$ and $\tilde{Y} = \tilde{q}_{max} - \tilde{q}_{med}$, where

$$\tilde{q}_{max} = max(|q_{1,2}|, |q_{2,3}|, |q_{3,1}|)$$
$$\tilde{q}_{med} = med(|q_{1,2}|, |q_{2,3}|, |q_{3,1}|)$$
$$\tilde{q}_{min} = S\,min(|q_{1,2}|, |q_{2,3}|, |q_{3,1}|) \quad (2)$$
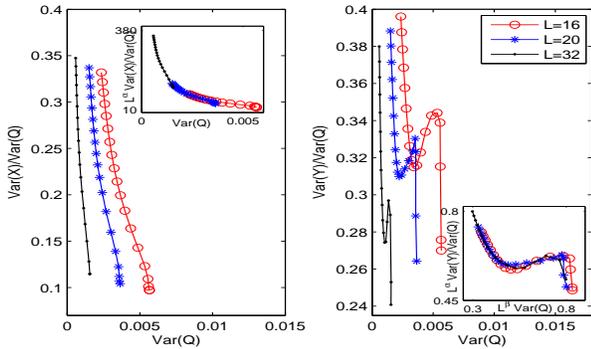
a simple numerical simulation sees that the plots are

FIG. 1: Normalized variances of the two random variables X (left) and Y (right) as a function of $Var(Q)$ for 2D $+/-1$. The left inset shows the scaling law $L^{\alpha}Var(X)/Var(Q)$ for $\alpha = 2$ and the right inset the scaling law $L^{\alpha}Var(Y)/Var(Q)$ as a function of $L^{\beta}Var(Q)$ for $\alpha = 0.22$ and $\beta = 1.8$. In both cases the scaled normalized variances are L-independent.

aligned along the lines

$$\rho_{\tilde{X}}(x) = \begin{cases} -\frac{9}{2}x + 3 & \text{for } 0 \le x \le \frac{2}{3} \\ 0 & \text{for } \frac{2}{3} < x \le 1 \end{cases}$$
$$\rho_{\tilde{Y}}(y) = -2y + 2 \tag{3}$$

as also a direct analytical computation shows in the large volume limit. Clearly the previous formulas cannot satisfy the RSB ultrametric relation of Eq. (1) .

In conclusion the argument of [1] cannot be used to weaken the result presented in [2] which was based mainly on identifying the scale law that governs the approach to ultrametricity. The same method used in [2] and properly applied to the two dimensional case reveals in fact the expected lack of RSB picture at positive temperature. The numerical result presented in [1] is interesting only because it shows that for small volumes some ultrametric features may persist if the system is investigated outside but close enough to the spin glass region. The *apparent* ultrametricity observed in [1] is due to the closeness of their simulation to the critical line in the plane (T, d). Moving away from the line, for instance performing the analysis in $(0, 1)$ the disappearance of ultrametricity is immediately seen also in finite volume systems.

[1] T.Jorg and F.Krzakala *Comment on "Ultra-metricity in the Edwards-Anderson Model"* http://arxiv.org/abs/0709.0894
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