SPIN GLASS IDENTITIES
AND THE NISHIMORI LINE

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Abstract

For a general spin glass model with asymmetric couplings we prove a family of identities involving expectations of generalized overlaps and magnetizations in the quenched state. Those identities holds pointwise in the Nishimori line and are reached at the rate of the inverse volume while, in the general case, they can be proved in integral average.

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1 Introduction and results

Overlap identities have played and continue to play a central role in spin glass statistical mechanics since the appearance of the Parisi solution [MPV] of the Sherrington Kirkpatrick model [SK]. The replica symmetry breaking theory contains indeed, as a built-in ansatz property, a family of identity for the overlap expectations that have been since then classified as replica equivalence and ultrametricity factorization properties. While the first is now largely understood for both mean field and short range finite dimensional models [CGi, CGi2] the second is still an open conjecture even within the mean field cases. The ideas to obtain the rigorous proof of the identities trace back to the papers [AC, GG] where the invariance property of stochastic stability and the role of the energy fluctuations for the spin glass quenched measure were introduced. The mentioned properties state a striking feature for the quenched measure on overlap expectations: the overlap moments do obey sum rules that reduce the distributional degrees of freedom and, at least in the mean field case, it is expected that the entire distribution be identified by the simple overlap distribution. Those relations are expected to hold everywhere but on isolated singularities and they can be in fact rigorously proved in β Riemann integral average [CGi2, T, B].

We present in this paper a new family of identities in terms of generalized overlap and magnetization that hold when the interactions are not centered which are proved in integral average. Moreover we find the remarkable result that in the Nishimori line they hold everywhere with respect to the parameters and are reached at the expected rate of the inverse volume.

We can illustrate the results proved in this paper considering the Edwards-Anderson spin glass model with non symmetric Gaussian interactions of variance Δ^2 and non zero average μ. Considering the quenched measure over multiple copies of the system subject to the same disorder (for a precise definition see Section 2) the *link overlaps* and the *link magnetizations* fulfill an infinite family of identities when integrated in the inverse temperature β over arbitrary intervals. Those identities at the lowest moment read, for

all μ and in the thermodynamic limit:

$$\int_{\beta_1}^{\beta_2} d\beta \ | \ \beta^2 < -2q_{1,2}^2 + 8q_{1,2}q_{2,3} - 6q_{1,2}q_{3,4} >$$
$$+ \quad \beta < 4q_{1,2}m_3 - 4q_{1,2}m_1 >$$
$$+ \quad < m_1^2 - m_1m_2 > | = 0 \tag{1.1}$$

$$\int_{\beta_1}^{\beta_2} d\beta \ | \ \beta^2(< q_{1,2}^2 > -6 < q_{1,2} >< q_{2,3} > +6 < q_{1,2}q_{3,4} > - < q_{1,2} >^2)$$
$$+ \quad \beta(2 < q_{1,2}m_1 > -4 < q_{1,2}m_3 > +2 < q_{1,2} >< m_1 >)$$
$$+ \quad < m_1m_2 > - < m_1 >^2 \ | = 0 \tag{1.2}$$

Since we prove that the magnetization has vanishing fluctuation in μ-average in the infinite volume limit

$$\int_{\mu_1}^{\mu_2} d\mu(< m^2 > - < m >^2) = 0 \tag{1.3}$$

then, when integrated over arbitrary boxes in the (β, μ) plane, the previous identities become

$$\int_{\mu_1}^{\mu_2} d\mu \int_{\beta_1}^{\beta_2} d\beta |\beta^2 < -2q_{1,2}^2 + 8q_{1,2}q_{2,3} - 6q_{1,2}q_{3,4} > | = 0 \tag{1.4}$$

$$\int_{\mu_1}^{\mu_2} d\mu \int_{\beta_1}^{\beta_2} d\beta |\beta^2(< q_{1,2}^2 > -6 < q_{1,2} >< q_{2,3} > +6 < q_{1,2}q_{3,4} > - < q_{1,2} >^2)| = 0 \ , \tag{1.5}$$

thus reducing to the standard ones [Pa, G, AC].

In the Nishimori line [N1, N2],

$$\beta\Delta^2 = \mu \tag{1.6}$$

the identities (1.1) and (1.2) hold pointwise in β and μ and the identities (1.4) and (1.5) hold when integrated over μ on arbitrary intervals.

The plan of the paper is the following: in the next Section we define the general class of models for which our result apply and we set the notations. In Section 3 we state the theorems, which are then proved in Section 4.

2 Definitions

We consider a disordered model of Ising configurations $\sigma_n = \pm 1$, $n \in \Lambda \subset \mathbb{Z}^d$ for some d-parallelepiped Λ of volume $|\Lambda|$. We denote Σ_Λ the set of all $\sigma = \{\sigma_n\}_{n\in\Lambda}$, and $|\Sigma_\Lambda| = 2^{|\Lambda|}$. In the sequel the following definitions will be used.

1. *Hamiltonian.*

   For every $\Lambda \subset \mathbb{Z}^d$ let $\{H_\Lambda(\sigma)\}_{\sigma\in\Sigma_N}$ be a family of $2^{|\Lambda|}$ *translation invariant (in distribution) Gaussian* random variables defined, in analogy with [RU], according to the general representation

   $$H_\Lambda(\sigma) = -\sum_{X\subset\Lambda} J_X \sigma_X \tag{2.7}$$

   where

   $$\sigma_X = \prod_{i\in X} \sigma_i , \tag{2.8}$$

   $(\sigma_\emptyset = 0)$ and the J's are independent Gaussian variables with mean

   $$\mathrm{Av}(J_X) = \mu_X , \tag{2.9}$$

   and variance

   $$\mathrm{Av}((J_X - \mu_X)^2) = \Delta_X^2 . \tag{2.10}$$

2. *Average and Covariance matrix.*

   The Hamiltonian has average

   $$\mathcal{B}_\Lambda(\sigma) := \mathrm{Av}(H_\Lambda(\sigma)) = \sum_{X\subset\Lambda} \mu_X \sigma_X , \tag{2.11}$$

   and covariance matrix

   $$\begin{aligned}
   \mathcal{C}_\Lambda(\sigma,\tau) &:= \mathrm{Av}\left([H_\Lambda(\sigma) - \mathcal{B}_\Lambda(\sigma)][H_\Lambda(\tau)) - \mathcal{B}_\Lambda(\tau)]\right) \\
   &= \sum_{X\subset\Lambda} \Delta_X^2 \sigma_X \tau_X .
   \end{aligned} \tag{2.12}$$

   By the triangular inequality

   $$|\mathcal{B}_\Lambda(\sigma)| \le \sum_{X\subset\Lambda} |\mu_X| \tag{2.13}$$

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for all σ and by the Schwarz inequality

$$|\mathcal{C}_\Lambda(\sigma,\tau)| \;\leq\; \sqrt{\mathcal{C}_\Lambda(\sigma,\sigma)}\sqrt{\mathcal{C}_\Lambda(\tau,\tau)} \;=\; \sum_{X\subset\Lambda}\Delta_X^2 \tag{2.14}$$

for all σ and τ.

3. *Thermodynamic Stability.*

The Hamiltonian (2.7) is thermodynamically stable if there exist constants $\bar{c}$ such that

$$\sup_{\Lambda\subset\mathbb{Z}^d}\frac{1}{|\Lambda|}\sum_{X\subset\Lambda}|\mu_X| \;\leq\; \bar{c} \;<\; \infty$$

$$\sup_{\Lambda\subset\mathbb{Z}^d}\frac{1}{|\Lambda|}\sum_{X\subset\Lambda}\Delta_X^2 \;\leq\; \bar{c} \;<\; \infty\,. \tag{2.15}$$

Thanks to the relations (2.13) and (2.14) a thermodynamically stable model fulfills the bound

$$\mathcal{B}_\Lambda(\sigma) \;\leq\; \bar{c}\,|\Lambda|$$

$$\mathcal{C}_\Lambda(\sigma,\tau) \;\leq\; \bar{c}\,|\Lambda| \tag{2.16}$$

and has an order 1 normalized mean and covariance

$$b_\Lambda(\sigma) \;:=\; \frac{1}{|\Lambda|}\mathcal{B}_\Lambda(\sigma)$$

$$c_\Lambda(\sigma,\tau) \;:=\; \frac{1}{|\Lambda|}\mathcal{C}_\Lambda(\sigma,\tau)\,. \tag{2.17}$$

4. *Random partition function.*

$$\mathcal{Z}(\beta) \;:=\; \sum_{\sigma\in\Sigma_\Lambda}e^{-\beta H_\Lambda(\sigma)}\,. \tag{2.18}$$

5. *Random free energy.*

$$-\beta\mathcal{F}(\beta) \;:=\; \mathcal{A}(\beta) \;:=\; \ln\mathcal{Z}(\beta)\,. \tag{2.19}$$

6. *Random internal energy.*

$$\mathcal{U}(\beta) \;:=\; \frac{\sum_{\sigma\in\Sigma_\Lambda}H_\Lambda(\sigma)e^{-\beta H_\Lambda(\sigma)}}{\sum_{\sigma\in\Sigma_\Lambda}e^{-\beta H_\Lambda(\sigma)}}\,. \tag{2.20}$$

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7. *Quenched free energy.*

$$-\beta F(\beta) \;:=\; A(\beta) \;:=\; \mathrm{Av}\,(\mathcal{A}(\beta))\;. \tag{2.21}$$

8. *R-product random Gibbs-Boltzmann state.*

$$\Omega(-) \;:=\; \sum_{\sigma^{(1)},\ldots,\sigma^{(R)}} (-)\,\frac{e^{-\beta[H_\Lambda(\sigma^{(1)})+\cdots+H_\Lambda(\sigma^{(R)})]}}{[\mathcal{Z}(\beta)]^R}\;. \tag{2.22}$$

9. *Quenched equilibrium state.*

$$< - > := \mathrm{Av}\,(\Omega(-))\;. \tag{2.23}$$

10. *Observables.*

For any smooth bounded function $G(b_\Lambda, c_\Lambda)$ (without loss of generality we consider $|G| \leq 1$ and no assumption of permutation invariance on G is made) of the mean and covariance matrix entries we introduce the random (with respect to $< - >$) R-dimensional vector of elements $\{m_k\}$ (called *generalized magnetization*) and the $R \times R$ matrix of elements $\{q_{k,l}\}$ (called *generalized overlap*) by the formula

$$< G(m, q) > \;:=\; \mathrm{Av}\,(\Omega(G(b_\Lambda, c_\Lambda)))\;. \tag{2.24}$$

E.g.: $G(b_\Lambda, c_\Lambda) = b_\Lambda(\sigma^1)c_\Lambda(\sigma^{(1)}, \sigma^{(2)})c_\Lambda(\sigma^{(2)}, \sigma^{(3)})$

$$< m_1 q_{1,2} q_{2,3} > = \mathrm{Av}\left(\sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} b_\Lambda(\sigma^1)c_\Lambda(\sigma^{(1)}, \sigma^{(2)})c_\Lambda(\sigma^{(2)}, \sigma^{(3)})\,\frac{e^{-\beta[\sum_{i=1}^{3} H_\Lambda(\ \sigma^{(i)})]}}{[\mathcal{Z}(\beta)]^3}\right) \tag{2.25}$$

3 Theorems

To state our results we introduce the random variable J' defined by

$$J_X = J'_X + \mu_X\;,$$

and deform uniformly the averages μ_X with a parameter μ defined by

$$\mu\mu'_X = \mu_X\;,$$

in such a way that

$$J_X = J_X' + \mu \mu_X' \; .$$

Our results can be summarized in the following theorems.

3.1   Identities in β-average

For every observable G of the kind considered in the previous Section, we define

$$f_1(\beta, \mu) \;\; = \;\; \sum_{l=1}^{R} < (m_l - m_{R+1}) \, G > + \tag{3.26}$$

$$-\beta \left(< \sum_{\substack{k,l=1 \\ k \neq l}}^{R} G \, q_{l,k} - 2RG \sum_{l=1}^{R} q_{l,R+1} + R(R+1) G \, q_{R+1,R+2} > \right)$$

and

$$f_2(\beta, \mu) \;\; = \;\; < m_{R+1} G > - < m_1 >< G > + \tag{3.27}$$

$$-\beta \left(\sum_{k=1}^{R+1} < G \, q_{k,R+1} > -(R+1) < G \, q_{R+1,R+2} > - < G > (< q_{1,1} > - < q_{1,2} >) \right)$$

We then have the following

Theorem 1 *The quenched equilibrium state of a thermodynamically stable Hamiltonian fulfills, for every observable G and every temperature interval $[\beta_1, \beta_2]$ the following identities in the thermodynamic limit*

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \int_{\beta_1}^{\beta_2} d\beta \; |f_1(\beta, \mu)| \;\; = \;\; 0 \tag{3.28}$$

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \int_{\beta_1}^{\beta_2} d\beta \; |f_2(\beta, \mu)| \;\; = \;\; 0 \tag{3.29}$$

Remark 1 *The two previous relations when applied to $G(m,q) = \beta q_{1,2} + m_1$ and $R = 2$ yields the identities mentioned in the introduction, formulae (1.1) and (1.2) - here it is assumed that $q_{1,1} = 1$, as it happens for the Edwards Anderson model.*

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3.2 Identities in (β, μ)-average

We also define

$$g_1(\beta, \mu) = -\beta \left(< \sum_{\substack{k,l=1 \\ k \neq l}}^{R} G\, q_{l,k} - 2RG \sum_{l=1}^{R} q_{l,R+1} + R(R+1)G\, q_{R+1,R+2} > \right) \qquad (3.30)$$

and

$$g_2(\beta, \mu) = -\beta \left(\sum_{k=1}^{R+1} < G\, q_{k,R+1} > -(R+1) < G\, q_{R+1,R+2} > - < G > (< q_{1,1} > - < q_{1,2} >) \right) \qquad (3.31)$$

Theorem 2 *The quenched equilibrium state of a thermodynamically stable Hamiltonian fulfills, for every observable G and every set $[\beta_1, \beta_2] \times [\mu_1, \mu_2]$ the following identities in the thermodynamic limit*

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \int_{\mu_1}^{\mu_2} d\mu \int_{\beta_1}^{\beta_2} d\beta \, |g_1(\beta, \mu)| = 0 \qquad (3.32)$$

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \int_{\mu_1}^{\mu_2} d\mu \int_{\beta_1}^{\beta_2} d\beta \, |g_2(\beta, \mu)| = 0 \qquad (3.33)$$

Remark 2 *Two similar families of identities have been proved in [CGi2] for the centered case $\mathrm{Av}\,(J_X) = 0$ for all X. This theorem generalizes the old result and reduces to it when the observable G doesn't depend on m.*

3.3 Identities pointwise

Theorem 3 *In the space of parameters $(\mu_X, \Delta_X^2)_{X \in \Lambda}$ there exists a region called the Nishimori manifold*

$$\mu_X = \beta \Delta_X^2 \qquad (3.34)$$

where the identities of Theorem (1) hold pointwise, namely

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} f_1(\beta, \mu) = 0 \tag{3.35}$$

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} f_2(\beta, \mu) = 0 \tag{3.36}$$

4 Proofs

Theorem 1 is proved in the lemmas of subsections 4.1 and 4.2. The proof uses only elementary methods like concentration and classical inequalities. Along the same line Theorem 2 is proved in subsection 4.3. Theorem 3 is proved in subsection 4.4 making use of an exact computation on the Nishimori manifold.

Let $h(\sigma) = |\Lambda|^{-1} H_\Lambda(\sigma)$ denote the Hamiltonian per particle. We consider the quantity

$$\sum_{l=1}^{R} \left\{ < h(\sigma^{(l)}) \, G > - < h(\sigma^{(l)}) >< G > \right\} = \Delta_1 G + \Delta_2 G \tag{4.37}$$

where

$$\Delta_1 G = \sum_{l=1}^{R} \left\{ \mathrm{Av} \left(\Omega[h(\sigma^{(l)}) \, G] - \Omega[h(\sigma^{(l)})]\Omega[G] \right) \right\} \tag{4.38}$$

$$\Delta_2 G = \sum_{l=1}^{R} \left\{ \mathrm{Av} \left(\Omega[h(\sigma^{(l)})]\Omega[G] \right) - \mathrm{Av} \left(\Omega[h(\sigma^{(l)})] \right) \mathrm{Av} \left(\Omega[G] \right) \right\} \tag{4.39}$$

We are going to show that both $\Delta_1 G$ and $\Delta_2 G$ vanish (in β average) in the thermodynamic limit. This implies, by a simple application of integration by parts, the relations (3.28) and (3.29).

4.1 Stochastic Stability Bound

We follow the method of stochastic stability as developed in [CGi].

Lemma 4.1 *For any inverse temperature interval $[\beta_1, \beta_2]$ one has*

$$\int_{\beta_1}^{\beta_2} \mathrm{Av} \left(\Omega(h^2) - \Omega^2(h) \right) \, d\beta \leq \frac{\bar{c}(2 + \beta_1 + \beta_2)}{|\Lambda|} \tag{4.40}$$

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Proof.

The variance of the Hamiltonian per particle $h(\sigma)$ with respect to the Boltzmann state is nothing but (minus) the derivative of the average of $h(\sigma)$ up to a factor $1/|\Lambda|$, i.e.

$$\mathrm{Av}\left(\Omega(h^2) - \Omega^2(h)\right) = -\frac{1}{|\Lambda|}\frac{d}{d\beta}\mathrm{Av}\left(\Omega(h)\right) \tag{4.41}$$

Application of integration by parts (for its general form see formula (4.50) below) yields

$$
\begin{aligned}
\mathrm{Av}\left(\Omega(h)\right) &= \frac{1}{|\Lambda|}\mathrm{Av}\left(\sum_{X\subset\Lambda} J_X\Omega(\sigma_X)\right) \\
&= \frac{1}{|\Lambda|}\mathrm{Av}\left(\sum_{X\subset\Lambda}\mu_X\Omega(\sigma_X)\right) + \frac{1}{|\Lambda|}\sum_{X\subset\Lambda}\beta\Delta_X^2[1 - \mathrm{Av}\left(\Omega^2(\sigma_X)\right)] \\
&\leq (1+\beta)\bar{c}\,, \tag{4.42}
\end{aligned}
$$

where thermodynamic stability condition, Eq. (2.16), has been used in the last inequality. The lemma statement follows from integration of Eq. (4.41) over an arbitrary inverse temperature interval $[\beta_1, \beta_2]$ and the use of fundamental theorem of calculus together with the bound (4.42). $\qquad\square$

Lemma 4.2 *For every bounded observable G, see definition 10 of Section 2, we have that for every interval $[\beta_1, \beta_2]$ in the thermodynamic limit one has*

$$\int_{\beta_1}^{\beta_2} |\Delta_1 G|\, d\beta = 0 \tag{4.43}$$

Proof.

From the definition of $\Delta_1 G$, Eq.(4.38), we have

$$
\begin{aligned}
\int_{\beta_1}^{\beta_2} |\Delta_1 G|\, d\beta &\leq \int_{\beta_1}^{\beta_2} \sum_{l=1}^{R} |\mathrm{Av}\left(\Omega[h(\sigma^{(l)})\, G] - \Omega[h(\sigma^{(l)})]\Omega[G]\right)|\, d\beta \tag{4.44} \\
&\leq \int_{\beta_1}^{\beta_2} \sum_{l=1}^{R} \sqrt{\mathrm{Av}\left(\{\Omega[h(\sigma^{(l)})\, G] - \Omega[h(\sigma^{(l)})]\Omega[G]\}^2\right)}\, d\beta \tag{4.45} \\
&\leq R\int_{\beta_1}^{\beta_2} \sqrt{\mathrm{Av}\left(\Omega[h^2(\sigma)] - \Omega^2[h(\sigma)]\right)}\, d\beta \tag{4.46} \\
&\leq R\sqrt{\beta_2 - \beta_1}\sqrt{\int_{\beta_1}^{\beta_2} \mathrm{Av}\left(\Omega[h^2(\sigma)] - \Omega^2[h(\sigma)]\right)\, d\beta} \tag{4.47} \\
&\leq R\sqrt{\beta_2 - \beta_1}\sqrt{\frac{\bar{c}(2 + \beta_1 + \beta_2)}{|\Lambda|}} \tag{4.48}
\end{aligned}
$$

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where (4.44) follows from triangular inequality, (4.45) is obtained by applying Jensen inequality on the measure $\mathrm{Av}\,(-)$, (4.46) comes from application of Schwarz inequality to the measure $\Omega(-)$ and boundedness of G, (4.47) is again Jensen inequality on the measure $\frac{1}{\beta_2 - \beta_1} \int_{\beta_1}^{\beta_2} (-) d\beta$ and finally (4.48) comes from lemma 4.1. $\qquad\square$

Remark 3 *The previous lemma is related to a general property of disordered systems which is known as stochastic stability (see [AC, CGi]). It says that the equilibrium state in a spin glass model is invariant under a suitable class of perturbation in all temperature intervals of continuity. The result presented in ([CGi]) for the case of zero average couplings holds with the absolute value outside the integral in beta with a vanishing rate of the inverse volume. Here instead we got the stronger result for the absolute value inside the integral but with a weaker vanishing rate of the square root inverse volume.*

Lemma 4.3 *The following expression holds:*

$$
\begin{aligned}
\Delta_1 G \;=\; & \sum_{l=1}^{R} < (m_l - m_{R+1})\, G > \\
& - \; \beta < G \left[\sum_{\substack{k,l=1 \\ k \neq l}}^{R} q_{l,k} - 2R \sum_{l=1}^{R} q_{l,R+1} + R(R+1)\, q_{R+1,R+2} \right] > .
\end{aligned}
\tag{4.49}
$$

Proof.

For each replica l $(1 \leq l \leq R)$, we evaluate separately the two terms in the sum of the right side of Eq. (4.38) by using the integration by parts (generalized Wick formula) for correlated Gaussian random variables, $x_1, x_2, \ldots, x_n$ with means $\mathrm{Av}\,(x_i)$ and covariances $\mathrm{Av}\,((x_i - \mathrm{Av}\,(x_i))(x_j - \mathrm{Av}\,(x_j)))$, namely

$$
\begin{aligned}
\mathrm{Av}\,(x_i\, \psi(x_1, \ldots, x_n)) \;=\; & \mathrm{Av}\,(x_i)\, \mathrm{Av}\,(\psi(x_1, \ldots, x_n)) \\
& + \sum_{j=1}^{n} \mathrm{Av}\,((x_i - \mathrm{Av}\,(x_i))(x_j - \mathrm{Av}\,(x_j)))\, \mathrm{Av}\left(\frac{\partial \psi(x_1, \ldots, x_n)}{\partial x_j} \right) .
\end{aligned}
\tag{4.50}
$$

It is convenient to denote by $p\,(R)$ the Gibbs-Boltzmann weight of R copies of the system

$$
p_R\,(\sigma^1, \ldots \sigma^R) \;=\; \frac{e^{-\beta \left[\sum_{k=1}^{R} H_\Lambda(\sigma^{(k)}) \right]}}{[\mathcal{Z}(\beta)]^R} ,
\tag{4.51}
$$

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so that we have

$$-\frac{1}{\beta}\frac{dp_R\left(\sigma^1,\ldots,\sigma^R\right)}{dH(\tau)} = p_R\left(\sigma^1,\ldots,\sigma^R\right)\left(\sum_{k=1}^{R}\delta_{\sigma^{(k)},\tau}\right) - R\,p_{R+1}\left(\sigma^1,\ldots,\sigma^R,\tau\right). \quad (4.52)$$

We obtain

$$\begin{aligned}
\mathrm{Av}\left(\Omega(h(\sigma^{(l)})\,G)\right) &= \frac{1}{|\Lambda|}\mathrm{Av}\left(\sum_{\sigma^{(1)},\ldots,\sigma^{(r)}} G\,H_\Lambda(\sigma^{(l)})\,p_R\left(\sigma^1,\ldots,\sigma^R\right)\right)\\
&= \mathrm{Av}\left(\sum_{\sigma^{(1)},\ldots,\sigma^{(r)}} G\,b_\Lambda(\sigma^{(l)})\,p_R\left(\sigma^1,\ldots,\sigma^R\right)\right)\\
&\quad + \mathrm{Av}\left(\sum_{\sigma^{(1)},\ldots,\sigma^{(r)}}\sum_{\tau} G\,c_\Lambda(\sigma^{(l)},\tau)\,\frac{dp_R\left(\sigma^1,\ldots,\sigma^R\right)}{dH(\tau)}\right) \quad (4.53)\\
&= <m_l\,G> -\beta\left[\sum_{k=1}^{R}<G\,q_{l,k}> -R<G\,q_{l,R+1}>\right] \quad (4.54)
\end{aligned}$$

where in (4.53) we made use of the integration by parts formula and (4.54) is obtained by (4.52). Analogously, the other term reads

$$\begin{aligned}
\mathrm{Av}\left(\Omega(h(\sigma^{(l)}))\,\Omega(G)\right) &= \frac{1}{|\Lambda|}\mathrm{Av}\left(\sum_{\sigma^{(l)}}\sum_{\tau^{(1)},\ldots,\tau^{(R)}} G\,H_\Lambda(\sigma^{(l)})\,p_{R+1}\left(\sigma^l,\tau_1,\ldots,\tau_R\right)\right)\\
&= \mathrm{Av}\left(\sum_{\sigma^{(l)}}\sum_{\tau^{(1)},\ldots,\tau^{(R)}} G\,b_\Lambda(\sigma^{(l)})\,p_{R+1}\left(\sigma^l,\tau_1,\ldots,\tau_R\right)\right)\\
&\quad + \mathrm{Av}\left(\sum_{\sigma^{(l)}}\sum_{\tau^{(1)},\ldots,\tau^{(R)}}\sum_{\gamma} G\,c_\Lambda(\sigma^{(l)},\gamma)\,\frac{dp_{R+1}\left(\sigma^l,\tau_1,\ldots,\tau_R\right)}{dH(\gamma)}\right)\\
&= <m_{R+1}\,G> -\beta\left[\sum_{k=1}^{R+1}<G\,q_{k,R+1}> -(R+1)<G\,q_{R+1,R+2}>\right]
\end{aligned}$$
$$(4.55)$$

Inserting the (4.54) and (4.55) in Eq. (4.38) and summing over l we obtain the expression (4.49). $\qquad\square$

4.2 Selfaveraging Bound

The selfaveraging of the free energy is a well established property of spin glass models. The vanishing of the fluctuations with respect to the disorder of the free energy can

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be obtained either by martingales arguments [PS, CGi2] or by concentration of measure [T, GT2]. Here we follow the second approach. Our formulation applies to both mean field and finite dimensional models and, for instance, includes the non summable interactions in finite dimensions [KS] and the p-spin mean field model as well as the REM and GREM models.

Lemma 4.4 *The disorder fluctuation of the free energy satisfies the following inequality: for all $x > 0$*

$$\mathbb{P}\left(|\mathcal{A} - \mathrm{Av}\left(\mathcal{A}\right)| \geq x\right) \ \leq \ 2\exp\left(-\frac{x^2}{2\bar{c}|\Lambda|}\right) \tag{4.56}$$

The free energy is then a self averaging quantity, i.e.

$$V(\mathcal{A}) \ = \ \mathrm{Av}\left(\mathcal{A}^2\right) - \mathrm{Av}\left(\mathcal{A}\right)^2 \ \leq \ 4\,\bar{c}\,\beta^2\,|\Lambda| \tag{4.57}$$

Proof. Consider an $s > 0$. By Markov inequality, one has

$$\begin{aligned}
\mathbb{P}\left\{\mathcal{A} - \mathrm{Av}\left(\mathcal{A}\right) \geq x\right\} \ &= \ \mathbb{P}\left\{\exp[s(\mathcal{A} - \mathrm{Av}\left(\mathcal{A}\right))] \geq \exp(sx)\right\} \\
&\leq \ \mathrm{Av}\left(\exp[s(\mathcal{A} - \mathrm{Av}\left(\mathcal{A}\right))]\right)\exp(-sx) \tag{4.58}
\end{aligned}$$

To bound the generating function

$$\mathrm{Av}\left(\exp[s(\mathcal{A} - \mathrm{Av}\left(\mathcal{A}\right))]\right) \tag{4.59}$$

one introduces, for a parameter $t \in [0,1]$, the following interpolating function:

$$\phi(t) = \ln Av_1\{\exp(s\ Av_2\{\ln Z(t)\})\}\,, \tag{4.60}$$

where $Av_1\{-\}$ and $Av_2\{-\}$ denote expectation with respect to two independent copies $X_1(\sigma)$ and $X_2(\sigma)$ of the random variable $X(\sigma)$, which is a *centered* Gaussian process with the same covariance as the Hamiltonian $H_\Lambda(\sigma)$, and the partition function $Z(t)$ is

$$Z(t) \ = \ \sum_{\sigma \in \Sigma_\Lambda} e^{-\beta\sqrt{t}X_1(\sigma) - \beta\sqrt{1-t}X_2(\sigma) - \beta\mathcal{B}_\Lambda(\sigma)}\,. \tag{4.61}$$

Indeed, since $H_\Lambda(\sigma) = X(\sigma) + \mathcal{B}_\Lambda(\sigma)$, it is immediate to verify (see definition (2.19)) that

$$\phi(0) = s\,\mathrm{Av}\left(\mathcal{A}\right)\,, \tag{4.62}$$

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and

$$\phi(1) = \ln \text{Av} \left(e^{s \, \mathcal{A}} \right) \, . \tag{4.63}$$

This implies that

$$\text{Av} \left(\exp[s(\mathcal{A} - \text{Av}\,(\mathcal{A}))] \right) = e^{\phi(1) - \phi(0)} = e^{\int_0^1 \phi'(t)dt} \tag{4.64}$$

On the other hand, the derivative with respect to t can be easily bounded. Defining

$$K(t) = \exp(s \, \text{Av}_2\{\ln Z(t)\}) \tag{4.65}$$

and

$$p^t(\sigma) = \frac{e^{-\beta\sqrt{t}X_1(\sigma) - \beta\sqrt{1-t}X_2(\sigma) - \beta\mathcal{B}_\Lambda(\sigma)}}{Z(t)} \tag{4.66}$$

one has

$$\phi'(t) = \frac{\text{Av}_1\left\{ K(t) \, s \, \text{Av}_2\left\{ \sum_\sigma p^t(\sigma) \left[\frac{1}{2\sqrt{t}}X_1(\sigma) - \frac{1}{2\sqrt{1-t}}X_2(\sigma) \right] \right\} \right\}}{\text{Av}_1\{K(t)\}} \tag{4.67}$$

Applying the integration by parts formula (4.50), a simple computation gives

$$\sum_\sigma \text{Av}_1\left\{ K(t) \, \text{Av}_2\left\{ p^t(\sigma)\frac{1}{\sqrt{t}} \, X_1(\sigma) \right\} \right\} = s\sum_{\sigma,\tau} \text{Av}_1\left\{ K(t) \, \mathcal{C}_\Lambda(\sigma,\tau) \, \text{Av}_2\{p^t(\tau)\} \, \text{Av}_2\{p^t(\sigma)\} \right\}$$

$$+ \quad \text{Av}_1\left\{ K(t) \, \text{Av}_2\left\{ \sum_\sigma \mathcal{C}_\Lambda(\sigma,\sigma)p^t(\sigma) \right\} \right\}$$

$$- \quad \text{Av}_1\left\{ K(t) \, \text{Av}_2\left\{ \sum_{\sigma,\tau} \mathcal{C}_\Lambda(\sigma,\tau)p^t(\sigma)p^t(\tau)) \right\} \right\}$$

and

$$\text{Av}_1\left\{ K(t) \, \text{Av}_2\left\{ \sum_\sigma p^t(\sigma)\frac{1}{\sqrt{1-t}} \, X_2(\sigma) \right\} \right\} = \text{Av}_1\left\{ K(t) \, \text{Av}_2\left\{ \sum_\sigma \mathcal{C}(\sigma,\sigma)_\Lambda p^t(\sigma) \right\} \right\}$$

$$- \quad \text{Av}_1\left\{ K(t) \, \text{Av}_2\left\{ \sum_{\sigma,\tau} \mathcal{C}(\sigma,\tau)_\Lambda p^t(\sigma)p^t(\tau)) \right\} \right\}$$

Taking the difference of the previous two expressions one finds

$$\phi'(t) = \frac{s^2}{2} \frac{\sum_{\sigma,\tau} \text{Av}_1\left\{ K(t) \, \mathcal{C}_\Lambda(\sigma,\tau) \, \text{Av}_2\{p^t(\tau)\} \, \text{Av}_2\{p^t(\sigma)\} \right\}}{\text{Av}_1\{K(t)\}} \tag{4.68}$$

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Using the thermodynamic stability condition (2.16), this yields

$$|\phi'(t)| \leq \frac{s^2}{2}|\Lambda|\bar{c} \tag{4.69}$$

from which it follows

$$\mathrm{Av}\left(\exp[s(\mathcal{A} - \mathrm{Av}\,(\mathcal{A}))]\right) \leq \exp\left(\frac{s^2}{2}|\Lambda|\bar{c}\right) \tag{4.70}$$

Inserting this bound into the inequality (4.58) and optimizing over s one finally obtains

$$\mathbb{P}\,(\mathcal{A} - \mathrm{Av}\,(\mathcal{A}) \geq x) \ \leq \ \exp\left(-\frac{x^2}{2\bar{c}|\Lambda|}\right) \tag{4.71}$$

The proof of inequality (4.56) is completed by observing that one can repeat a similar computation for $\mathbb{P}\,(\mathcal{A} - \mathrm{Av}\,(\mathcal{A}) \leq -x)$. The result for the variance (4.57) is then immediately proved using the identity

$$\mathrm{Av}\left((\mathcal{A} - \mathrm{Av}\,(\mathcal{A}))^2\right) = 2\int_0^\infty x\,\mathbb{P}(|\mathcal{A} - \mathrm{Av}\,(\mathcal{A})| \geq x)\,dx \tag{4.72}$$

$\square$

Lemma 4.5 *The internal energy is self averaging almost everywhere in β, i.e. defining $u = \mathcal{U}/|\Lambda|$ and $V(u) = \mathrm{Av}\,(u^2) - \mathrm{Av}\,(u)^2$ it holds in the thermodynamic limit*

$$\int_{\beta_1}^{\beta_2} V(u)\,d\beta \ \to \ 0 \tag{4.73}$$

Proof.

The result is obtained in two steps which use general theorems of measure theory. First from lemma 4.4 we obtain the convergence to zero almost everywhere (in β) of the variance of the internal energy, then thanks to a bound on the variance of the internal energy we apply the Lebesgue dominated convergence theorem which gives the lemma statement. The sequence of convex functions $\mathcal{A}(\beta)/|\Lambda|$ converges a.e. (in J) to the limiting value $a(\beta)$ of its average and the convergence is self averaging in the sense of lemma 4.4. By general convexity arguments [RU] it follows that the sequence of the derivatives $\mathcal{A}'(\beta)/|\Lambda|$ converges to $u(\beta) = a'(\beta)$ almost everywhere in β and also that the convergence is self averaging. In fact the vanishing of the variance of a sequence of convex functions is

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inherited, in all points in which the derivative exists (which is almost everywhere for a convex function), to the sequence of its derivatives (see [S, OTW]). From lemma 4.4 we have then

$$V(u) \;\to\; 0 \quad \beta - a.e. \tag{4.74}$$

In order to obtain the convergence in β-average we use the Lebesgue dominated convergence theorem. In fact we prove that the sequence of variances of u is uniformly bounded (in every interval $[\beta_1, \beta_2]$) by an integrable function of β. A lengthy but simple computation which uses again integration by parts gives

$$\begin{aligned}
\mathrm{Av}\left(\mathcal{U}^2\right) \;=\;& \mathrm{Av}\left(\sum_{X,Y \subset \Lambda} J_X J_Y \Omega(\sigma_X)\Omega(\sigma_Y)\right) & (4.75) \\
=\;& \sum_{X,Y \subset \Lambda} \mu_X \mu_Y \Omega(\sigma_X)\Omega(\sigma_Y) + \sum_{X \subset \Lambda} \Delta_X^2 \Omega^2(\sigma_X) \\
+\;& 2\beta \sum_{X,Y \subset \Lambda} \mu_X \Delta_Y^2 \left[\Omega(\sigma_X \sigma_Y)\Omega(\sigma_Y) + \Omega(\sigma_X) - 2\Omega(\sigma_X)\Omega^2(\sigma_Y)\right] \\
+\;& \beta^2 \sum_{X,Y \subset \Lambda} \Delta_X^2 \Delta_Y^2 \mathrm{Av}\left[1 - \Omega^2(\sigma_X) - \Omega^2(\sigma_Y) + 6\Omega^2(\sigma_X)\Omega^2(\sigma_Y) + \right. \\
& \left. -6\Omega(\sigma_X)\Omega(\sigma_Y)\Omega(\sigma_X \sigma_Y) + \Omega^2(\sigma_X \sigma_Y)\right] & (4.76)
\end{aligned}$$

from which

$$V(u) \;\leq\; \mathrm{Av}\left(\mathcal{U}^2\right) \;\leq\; |\Lambda|^2 \bar{c}^2 (2 + 4\beta + 14\beta^2). \tag{4.77}$$

From this follows (4.73). $\qquad\qquad\square$

Lemma 4.6 *For every bounded observable G, see definition 10 of Section 2, we have that for every interval $[\beta_1, \beta_2]$ in the thermodynamic limit*

$$\int_{\beta_1}^{\beta_2} |\Delta_2 G| \, d\beta \;=\; 0 \tag{4.78}$$

Proof.

From the definition of $\Delta_2 G$, Eq.(4.39), we have

$$\int_{\beta_1}^{\beta_2} |\Delta_2 G| \, d\beta \;\leq\; \int_{\beta_1}^{\beta_2} \sum_{l=1}^{R} \left| \mathrm{Av}\left(\Omega[h(\sigma^{(l)})]\Omega[G]\right) - \mathrm{Av}\left(\Omega[h(\sigma^{(l)})]\right) \mathrm{Av}\left(\Omega[G]\right) \right| \, d\beta \tag{4.79}$$

$$\leq\; \int_{\beta_1}^{\beta_2} \sum_{l=1}^{R} \sqrt{\mathrm{Av}\left(\Omega[h^2(\sigma^{(l)})]\right) - \left(\mathrm{Av}\left(\Omega[h(\sigma^{(l)})]\right)\right)^2} \, d\beta \tag{4.80}$$

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$$\leq \; R \int_{\beta_1}^{\beta_2} \sqrt{V(u)} \, d\beta \tag{4.81}$$

$$\leq \; R\sqrt{\beta_2 - \beta_1} \sqrt{\int_{\beta_1}^{\beta_2} V(u) \, d\beta} \tag{4.82}$$

where (4.79) follows from triangular inequality, (4.80) is obtained by applying Schwarz inequality to the measure $\mathrm{Av}\,(-)$ and boundedness of G, (4.82) is Jensen inequality on the measure $\frac{1}{\beta_2 - \beta_1} \int_{\beta_1}^{\beta_2} (-) d\beta$. The statement (4.78) follows then using the result of the previous lemma. □

Lemma 4.7 *The following expression holds:*

$$\Delta_2 G \;=\; \sum_{l=1}^{R} (< m_{R+1} G > - < m_l >< G >) + \tag{4.83}$$
$$-\beta R \left[\sum_{k=1}^{R+1} < G \, q_{k,R+1} > -(R+1) < G \, q_{R+1,R+2} > - < G > (< q_{1,1} > - < q_{1,2} >) \right].$$

Proof. In order to obtain $\Delta_2 G$ we are left with the explicit evaluation of the other term in (4.39) which simply gives

$$\mathrm{Av}\left(\Omega(h(\sigma^{(l)}))\right) \mathrm{Av}\left(\Omega(G)\right) \;=\; \frac{1}{|\Lambda|} \mathrm{Av}\left(\sum_{\sigma^{(l)}} H_\Lambda(\sigma^{(l)}) \, p_1(\sigma) \right) < G >$$
$$=\; \mathrm{Av}\left(\sum_{\sigma^{(l)}} b_\Lambda(\sigma^{(l)}) \, p_1(\sigma) \right) < G >$$
$$+\mathrm{Av}\left(\sum_{\sigma^{(l)}} \sum_{\gamma} c_\Lambda(\sigma^{(l)},\gamma) \, \frac{dp_1(\sigma)}{dH_\Lambda(\gamma)} \right) < G >$$
$$=\; < m_l >< G > -\beta < G > [< q_{1,1} > - < q_{1,2} >] \tag{4.84}$$

Inserting the (4.55) and (4.84) in Eq. (4.39) and summing over l we obtain the (4.83). □

4.3 Vanishing fluctuations of the generalized magnetization

Lemma 4.8 *For every interval $[\mu_1, \mu_2]$, in the thermodynamic limit*

$$\int_{\mu_1}^{\mu_2} d\mu(< m^2 > - < m >^2) = 0 \tag{4.85}$$

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Proof. The proof that the generalized magnetization has vanishing fluctuation follows the strategy that has been pursued so far to control fluctuation of the internal energy. We have

$$< m^2 > - < m >^2 = \text{Av}\left(\Omega(b_\Lambda(\sigma)^2) - (\Omega(b_\Lambda(\sigma))^2)\right) \tag{4.86}$$

$$+ \text{Av}\left((\Omega(b_\Lambda(\sigma))^2) - (\text{Av}(\Omega(b_\Lambda(\sigma))))^2\right) \tag{4.87}$$

and we observe that generalized magnetization is related to the pressure by

$$\frac{\partial}{\partial \mu}\left(\frac{\mathcal{A}}{|\Lambda|}\right) = \frac{\beta}{\mu}\frac{1}{|\Lambda|}\Omega(\sum_{X\in\Lambda}\mu_X\sigma_X) = \frac{\beta}{\mu}\,\Omega(b_\Lambda(\sigma)) \tag{4.88}$$

The fluctuations w.r.t. the Gibbs state (r.h.s of Eq.(4.86)) are easily controlled by a stochastic stability argument:

$$\int_{\mu_1}^{\mu_2} d\mu \text{Av}\left(\Omega(b_\Lambda(\sigma)^2) - (\Omega(b_\Lambda(\sigma))^2)\right) = \frac{1}{|\Lambda|}\int_{\mu_1}^{\mu_2}d\mu\,\frac{\mu^2}{\beta^2}\,\text{Av}\left(\frac{\partial}{\partial\mu}\frac{\beta}{\mu}\Omega(b_\Lambda(\sigma))\right), \tag{4.89}$$

where the right hand side can be bounded, integrating by parts in μ by $\beta^{-1}3(\mu_2 - \mu_1)$.

The fluctuations w.r.t. the disorder (Eq.(4.87)) are bounded by the same argument of lemma 4.5. Indeed from self-averaging of the pressure per particle

$$V\left(\frac{\mathcal{A}}{|\Lambda|}\right) \le \frac{c}{|\Lambda|} \to 0 \,,$$

and convexity of finite volume pressure

$$\frac{\partial^2}{\partial\mu^2}\frac{\mathcal{A}}{|\Lambda|} = |\Lambda|[\Omega(b_\Lambda(\sigma)^2) - (\Omega(b_\Lambda(\sigma))^2] \ge 0$$

one deduces that also the sequence of derivatives (4.88) is self-averaging in μ-average. Hence

$$\int_{\mu_1}^{\mu_2} d\mu\left[\text{Av}\left((\Omega(b_\Lambda(\sigma))^2) - (\text{Av}(\Omega(b_\Lambda(\sigma))))^2\right] = \int_{\mu_1}^{\mu_2}d\mu V\left(\frac{\partial}{\partial\mu}\left(\frac{\mathcal{A}}{|\Lambda|}\right)\right) \to 0 \tag{4.90}$$

Combining together (4.89) and (4.90) complete the proof of the lemma. $\square$

4.4 Nishimori manifold

Our strategy here is to prove the vanishing of fluctuations of the Hamiltonian per particle $h(\sigma)$. This implies, following the same reasoning of the previous sections, the vanishing

of correlations with a generic bounded function G using the Schwarz inequality:

$$| < hG > - < h >< G > | \leq \sqrt{< h^2 > - < h >^2}\sqrt{< G^2 > - < G >^2} . \qquad (4.91)$$

Lemma 4.9 *On the Nishimori manifold*

$$\mu_X = \beta \Delta_X^2 \qquad (4.92)$$

the random internal energy per particle is selfaveraging. More precisely the following result holds:

$$< h^2 > - < h >^2 \leq \frac{\bar{c}}{|\Lambda|} \qquad (4.93)$$

Proof. This is proved by an explicit computation of both terms in formula (4.93). Let us first consider

$$\mathrm{Av}\left(\Omega(J_X\sigma_X)\right) .$$

We first write the definition explicitly:

$$\mathrm{Av}\left(\Omega(J_X\sigma_X)\right) = \int_{-\infty}^{\infty} \prod_Y \left(dJ_Y\frac{1}{\sqrt{2\pi}\Delta_Y} \exp\left(-\frac{(J_Y - \mu_Y)^2}{2\Delta_Y^2}\right)\right) \cdot \frac{\sum_\sigma J_X\sigma_X e^{\beta\sum_Z J_Z\sigma_Z}}{\sum_\sigma e^{\beta\sum_Z J_Z\sigma_Z}} \qquad (4.94)$$

We apply the gauge transformation

$$J_X \to J_X\tau_X, \quad \sigma_i \to \sigma_i\tau_i,$$

for all $i \in \Lambda$ and $X \subset \Lambda$, where τ_i is a 'gauge' variable fixed to 1 or -1 at each $i \in X$ and $\tau_X = \prod_{i\in X} \tau_i$. This change of variables leaves the integral and sums in the above equation invariant. Then, only the J_Y in the exponent for the Gaussian weight changes:

$$\mathrm{Av}\left(\Omega(J_X\sigma_X)\right) \qquad (4.95)$$
$$= \int_{-\infty}^{\infty} \prod_Y \left(dJ_Y\frac{1}{\sqrt{2\pi}\Delta_Y} \exp\left(-\frac{(J_Y\tau_Y - \mu_Y)^2}{2\Delta_Y^2}\right)\right) \cdot \frac{\sum_\sigma J_X\sigma_X e^{\beta\sum_Z J_Z\sigma_Z}}{\sum_\sigma e^{\beta\sum_Z J_Z\sigma_Z}}$$
$$= \int_{-\infty}^{\infty} \prod_Y \left(dJ_Y\frac{1}{\sqrt{2\pi}\Delta_Y} \exp\left(-\frac{J_Y^2 + \mu_Y^2}{2\Delta_Y^2}\right)\right) \exp\left(\sum_Y \frac{J_Y\mu_Y\tau_Y}{\Delta_Y^2}\right) \cdot \frac{\sum_\sigma J_X\sigma_X e^{\beta\sum_Z J_Z\sigma_Z}}{\sum_\sigma e^{\beta\sum_Z J_Z\sigma_Z}}$$

Since this expression holds for any assignment of ± 1 to τ_i, we may sum it up over all possible $\{\tau_i\}_i$ and divide the result by $2^{|\Lambda|}$,

$$
\begin{aligned}
&\mathrm{Av}\left(\Omega(J_X \sigma_X)\right) \\
&= \frac{1}{2^{|\Lambda|}} \int_{-\infty}^{\infty} \prod_Y \left(dJ_Y \frac{1}{\sqrt{2\pi}\Delta_Y} \exp\left(-\frac{J_Y^2 + \mu_Y^2}{2\Delta_Y^2}\right) \right) \sum_\tau e^{\sum_Y J_Y \mu_Y \tau_Y / \Delta_Y^2} \\
&\quad \cdot \frac{\sum_\sigma J_X \sigma_X e^{\beta \sum_Z J_Z \sigma_Z}}{\sum_\sigma e^{\beta \sum_Z J_Z \sigma_Z}}
\end{aligned}
\tag{4.96}
$$

The sum over τ and the sum over σ in the denominator cancel each other for NL ($\beta = \mu_Y / \Delta_Y^2$), and we have a simplified expression

$$
\begin{aligned}
&\mathrm{Av}\left(\Omega(J_X \sigma_X)\right) \\
&= \frac{1}{2^{|\Lambda|}} \int_{-\infty}^{\infty} \prod_Y \left(dJ_Y \frac{1}{\sqrt{2\pi}\Delta_Y} \exp\left(-\frac{J_Y^2 + \mu_Y^2}{2\Delta_Y^2}\right) \right) \sum_\sigma J_X \sigma_X e^{\beta \sum_Z J_Z \sigma_Z} \\
&= \frac{1}{2^{|\Lambda|}} \sum_\sigma \int_{-\infty}^{\infty} \prod_Y \left(dJ_Y \frac{1}{\sqrt{2\pi}\Delta_Y} \exp\left(-\frac{J_Y^2 + \mu_Y^2}{2\Delta_Y^2}\right) \right) J_X \sigma_X e^{\sum_Z J_Z \sigma_Z \mu_Z / \Delta_Z^2}
\end{aligned}
\tag{4.97}
$$

For given $\{\sigma_i\}_i$, let us change the integral variable as $J_Y \to J_Y \sigma_Y$. Then σ disappears completely and the integral is just for the average of J_X

$$
\begin{aligned}
&\mathrm{Av}\left(\Omega(J_X \sigma_X)\right) \\
&= \frac{1}{2^{|\Lambda|}} \sum_\sigma \int_{-\infty}^{\infty} \prod_Y \left(dJ_Y \frac{1}{\sqrt{2\pi}\Delta_Y} \exp\left(-\frac{J_Y^2 + \mu_Y^2}{2\Delta_Y^2}\right) \right) J_X \, e^{\sum_Z J_Z \mu_Z / \Delta_Z^2} \\
&= \frac{1}{2^{|\Lambda|}} \cdot 2^{|\Lambda|} \cdot 1 \cdot \int_{-\infty}^{\infty} dJ_X \frac{1}{\sqrt{2\pi}\Delta_X} J_X \exp\left(-\frac{(J_X - \mu_X)^2}{2\Delta_X^2}\right) \\
&= \mathrm{Av}(J_X)
\end{aligned}
\tag{4.98}
$$

From the previous computation we obtain the final result for the quenched internal energy on the NL:

$$
\mathrm{Av}\left(\Omega(H_\Lambda(\sigma))\right) = \mathrm{Av}\left(\Omega\left(\sum_X J_X \sigma_X\right)\right) = \mathrm{Av}\left(\sum_X J_X\right) = \sum_X \mu_X
\tag{4.99}
$$

The other term in the variance (4.93) is evaluated similarly:

$$
\mathrm{Av}\left(\Omega(H_\Lambda(\sigma)^2)\right) = \mathrm{Av}\left(\Omega\left(\sum_{X,Y} J_X J_Y \sigma_X \sigma_Y\right)\right)
$$

$$
\begin{aligned}
&= \mathrm{Av}\left(\sum_{X,Y} J_X J_Y\right) \\
&= \sum_{X\neq Y} \mu_X \mu_Y + \sum_X (\mu_X^2 + \Delta_X^2) \\
&= \sum_{X,Y} \mu_X \mu_Y + \sum_X \Delta_X^2
\end{aligned}
\tag{4.100}
$$

Therefore, using (4.99) and (4.100) in the expression for the variance of h one finds

$$
\begin{aligned}
\langle h(\sigma)^2 \rangle - \langle h(\sigma) \rangle^2
&= \frac{1}{|\Lambda|^2} \mathrm{Av}\left(\Omega(H_\Lambda(\sigma)^2)\right) - \frac{1}{|\Lambda|^2}\{\mathrm{Av}\left(\Omega(H_\Lambda(\sigma))\right)\}^2 \\
&= \frac{1}{|\Lambda|^2} \sum_X \Delta_X^2 \leq \frac{\bar{c}}{|\Lambda|}
\end{aligned}
\tag{4.101}
$$

$\square$

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