The inclusion process: duality and correlation inequalities

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Abstract: We prove a *comparison inequality* between a system of independent random walkers and a system of random walkers which interact by attracting each other -a process which we call here the symmetric inclusion process (SIP). As an application, *correlation inequalities* for the SIP, as well as for a model of heat conduction, the so-called Brownian momentum process, are obtained. These inequalities are counterparts of the inequalities (in the opposite direction) for the symmetric exclusion process, showing that the SIP is a natural bosonic analogue of the symmetric exclusion process, which is fermionic. We discuss stationary measures of the SIP, and an asymmetric version that has the same stationary probability measures, as well as infinite non-translation invariant reversible measures. Finally, we consider a boundary driven version of the SIP for which we prove duality and correlation inequalities.

1 Introduction

In Liggett [11], Chapter VIII, proposition 1.7, a comparison inequality between independent symmetric random walkers and corresponding exclusion random walkers is obtained. This inequality plays a crucial role in the understanding of the exclusion

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process; it makes rigorous the intuitive picture that symmetric random walkers interacting by exclusion are more spread out than the corresponding independent walkers, as a consequence of the repulsive interaction (exclusion), or in more physical terms, because of the fermionic nature of the exclusion process. The comparison inequality is a key ingredient in the ergodic theory of the symmetric exclusion process, i.e., in the characterization of the invariant measures, and the measures which are in the course of time attracted to a given invariant measure. The comparison inequality has been generalized later on by Andjel [1], Liggett [12], and recently in the work of J. Borcea, P. Brändén and T.M. Liggett [2].

In the search of a natural conservative particle system where the opposite inequality holds, i.e., where the particles are *less spread out* than corresponding independent random walkers, it is natural to think of a "bosonic counterpart" of the exclusion process. Infact, such a process was introduced in [7] and [8] as the *dual* of the Brownian momentum process, a stochastic model of heat conduction (similar models of heat conduction were introduced in [3] and [6], see also [4] for the study of the structure function in a natural asymmetric version).

In the present paper we present a rigorous analysis of the "bosonic counterpart" of the exclusion process. We will call this process (as will be motivated by a Poisson clock representation) the "symmetric inclusion process" (SIP). In the SIP, jumps are performed according to independent random walks, and on top of that particles "invite" other particles to join their site (inclusion). For this process we prove the analogue of the comparison inequality for the symmetric exclusion process. From the comparison inequality, using the knowledge of the stationary measure and the self-duality property of the process, we deduce a series of correlation inequalities. Again, in going from exclusion to inclusion process the correlations turn from negative to positive. We remark however that these positive correlation inequalities are different from the ordinary preservation of positive correlations for monotone processes [10], because the SIP is not a monotone process. Since the SIP is dual to the heat conduction model it is immediate to extend those correlation inequalities to the Brownian momentum process.

We also introduce the non-equilibrium versions of the SIP. A first possibility is to consider the ASIP, i.e. asymmetry in the jump rates. In this case we show that the product probability measures of the symmetric case is still invariant; moreover we show that there exist other infinite inhomogeneous invariant measures, which are obtained from detailed balance and which might be the signal of possible condensation phenomena appearing in the model because of the attractive character of the interaction between particles. A second possibility to obtain a non equilibrium model is to consider the boundary driven version of SIP. In this case we prove duality of the process to a SIP model with absorbing boundary condition. We then deduce a correlation inequality, explaining and generalizing the positivity of the covariance in the non-equilibrium steady state of the heat conduction model in [7].

All the results will be also stated in the context of a family of SIP models, which are labeled by parameter $m \in \mathbb{N}$. As the SEP model can be generalized to the situation where there are at most $n \in \mathbb{N}$ particles per site (this corresponds to a quantum spin chain with SU(2) symmetry and spin value $j = n/2$), in the same way the SIP model can be extended to represent the situation of a quantum spin chain with SU(1,1) symmetry and spin value $k = m/4$.

2 Definition

Let S be a finite or a countable infinite set, and $p(x, y)$ a symmetric transition probability on S, i.e., $p(x, y) = p(y, x) \geq 0$, $\sum_y p(x, y) = 1$. We suppose that $p(x, y)$ is an irreducible (discrete-time) random walk transition probability.

The symmetric inclusion process associated to the transition kernel p is a process on $\Omega := \mathbb{N}^S$ with generator defined on the core of local functions by

$$Lf(\eta) = \sum_{x,y \in S} p(x, y) 2\eta_x (1 + 2\eta_y) \left(f(\eta^{x,y}) - f(\eta)\right) \tag{2.1}$$

where, for $\eta \in \Omega$, $\eta^{x,y}$ denotes the configuration obtained from η by removing one particle from x and putting it at y.

In [7] this model was introduced as the dual of a model of heat conduction, the so-called Brownian momentum process, see also [8], and [3] for generalized and or similar models of heat conduction.

The process with generator (2.1) can be interpreted as follows. Every particle has two exponential clocks: one clock -the so-called random walk clock- has rate 2, the other clock -the so-called inclusion clock- has rate 4. When the random walk clock of a particle at site $x \in S$ rings, the particle performs a random walk jump with probability $p(x, y)$ to site $y \in S$. When the inclusion process clock rings at site $y \in S$, with probability $p(y, x) = p(x, y)$ a particle from site $x \in S$ is selected and joins site y.

From this interpretation, we see that besides jumps of a system of independent random walkers, this system of particles has the tendency to bring particles together at the same site (inclusion), and can therefore be thought of as a "bosonic" counterpart of the symmetric exclusion process.

To make the analogy with the exclusion process even more transparent, in an exclusion process with at most n particles ($n \in \mathbb{N}$) per site ($SEP(n)$), the jump rate is $\eta_i(n - \eta_j)p(i, j)$. In section 7 we introduce the $SIP(m)$ model, which (up to an unessential re-scaling of all the rates by a factor 1/4) is obtained by changing the minus into a plus and choosing $n = m/2$. More precisely the jump rates of the $SIP(m)$ process are $2\eta_i(m + 2\eta_j)$ and if $m/2$ is an integer then the $SIP(m)$ is the analogue of the $SEP(m/2)$. We show that for the general class of $SIP(m)$ with $m \in \mathbb{N}$ we have the same correlation inequalities as for the SIP, duality, and explicit product stationary measures.

Notice that the rates in (2.1) are increasing both in the number of particles of the departure as of the arrival site of the jump (the rate is $p(x, y)(2\eta_x)(1 + 2\eta_y)$ for a particle to jump from x to y). Therefore, by the necessary and sufficient conditions of

[9], Theorem 2.21, the SIP is not a monotone process. It is also easy to see that due to the attraction between particles in the SIP, there cannot be a coupling that preserves the order of configurations, i.e., in any coupling starting from an ordered pair of configuration, the order will be lost in the course of time with positive probability.

2.1 Assumptions on the transition probability kernel

To prove some of our results we will need to make assumptions on the transition probability p. We define the associated continuous-time random walk transition probabilities (where the continuous walk jumps at rate 2 for later convenience),

$$p_t(x,y) = \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} e^{-2t} p^{(n)}(x,y) \tag{2.2}$$

where $p^{(n)}$ denotes the n^{th} power of the transition matrix p. Denote by $\mathbb{P}_{x,y}^{IRW}$ the probability measure on path space associated to two independent random walkers X_t, Y_t started at x, y and jumping according to (2.2) and by $\mathbb{P}_{x,y}^{SIP}$ the corresponding probability for two SIP walkers $X_t^{'}, Y_t^{'}$ jumping with the rates of generator (2.1).

The assumptions we will sometime make are:

- Assumption (A1)
$$\lim_{t\to\infty} \sup_{x,y} \mathbb{P}_{x,y}^{IRW}(X_t = Y_t) = 0 \tag{2.3}$$

- Assumption (A2)
$$\lim_{t\to\infty} \sup_{x,y} \mathbb{P}_{x,y}^{SIP}(X_t^{'} = Y_t^{'}) = 0 \tag{2.4}$$

The assumption (A1) amounts to requiring that for large $t > 0$, two independent random walkers walking according to the continuous time random walk probability (2.2) will be at the same place with vanishing probability. The assumption (A1) follows immediately if we restrict ourselves to the case where

$$\lim_{t\to\infty} \sup_{x,y} p_t(x,y) = 0 \tag{2.5}$$

since then

$$\lim_{t\to\infty} \sup_{x,y} \mathbb{P}_{x,y}^{IRW}(X_t = Y_t) = \lim_{t\to\infty} \sup_{x,y} \sum_{u\in S} p_t(x,u)p_t(y,u) = \lim_{t\to\infty} \sup_{x,y} p_{2t}(x,y) = 0 \tag{2.6}$$

Analogously, the assumption (A2) guarantees that two walkers evolving with the SIP dynamic will be typically at different positions. For example, in the translation invariant case $S = \mathbb{Z}^d$, $p(x,y) = p(0, y-x) =: \pi(x)$, this is automatically satisfied, as the difference walk $X_t^{'} - Y_t^{'}$ of two SIP particles is a random walk Z_t on $\mathbb{Z}^d$ with generator

$$L^Z f(z) = 8\pi(z)(f(0) - f(z)) + \sum_y 4\pi(y)(f(z+y) - f(z)) \tag{2.7}$$

which is clearly not positive recurrent.

Furthermore, assumption (A2) implies that any finite number of SIP particles will eventually be at different locations. This is made precise in Lemma 1 in section 5.

3 Comparison of the SIP with independent random walks

We will first consider the SIP process with a finite number of particle in subsection 3.1 and then state the comparison inequality in subsection 3.2.

3.1 The finite SIP

If we start the SIP with n particles at positions $x_1, \ldots, x_n \in S$, we can keep track of the labels of the particles. This gives then a continuous-time Markov chain on S^n with generator

$$\begin{aligned} \mathcal{L}_n f(x_1, \ldots, x_n) &= \sum_{i=1}^{n} \sum_{y \in S} 2p(x_i, y) \left(1 + 2\sum_{j=1}^{n} I(y = x_j)\right) (f(x^{x_i,y}) - f(x)) \\ &= \mathcal{L}_{1,n} f(x) + \mathcal{L}_{2,n} f(x) \end{aligned} \tag{3.1}$$

where $x^{x_i,y}$ denotes the n-tuple $(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$. Further, $\mathcal{L}_{1,n}$, resp. $\mathcal{L}_{2,n}$ denote the random walk resp. inclusion part of the generator and are defined as follows

$$\mathcal{L}_{1,n} f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{y \in S} 2p(x_i, y)(f(x^{x_i,y}) - f(x)) \tag{3.2}$$

$$\mathcal{L}_{2,n} f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} 4p(x_i, x_j)(f(x^{x_i,x_j}) - f(x)) \tag{3.3}$$

We call $T_n(t)$ the semigroup on functions $f : S^n \to \mathbb{R}$ associated to the generator (3.1), and $U_n(t)$ the semigroup of a system of independent continuous-time random walkers (jumping at rate 2), i.e., the semigroup associated to the generator $\mathcal{L}_{1,n}$.

3.2 Comparison inequality

From the description above, it is intuitively clear that in the SIP, particle tend to be less spread out than in a system of independent random walkers. The Theorem 1 below formalizes this intuition and is the analogue of a comparison inequality of the SEP ([11], Chapter VIII, Proposition 1.7).

To formulate it, we need the notion of a positive definite function. A function $f : S \times S \to \mathbb{R}$ is called positive definite if for all $\beta \in l_1(S)$,

$$\sum_{x,y} f(x, y)\beta(x)\beta(y) \geq 0$$

A function $f : S^n \to \mathbb{R}$ is called positive definite if it is positive definite in every pair of variables.

Theorem 1. *Let $f : S^n \to \mathbb{R}$ be positive definite. Then we have*

$$U_n(t)f(x) \le T_n(t)f(x) \tag{3.5}$$

for all $x \in S^n$.

Proof. Start with the decomposition (3.1) and use the symmetry of $p(x, y)$ to write

$$\begin{aligned}
& (\mathcal{L}_n f - \mathcal{L}_{1,n} f)(x) = (\mathcal{L}_{2,n} f)(x) \\
= & \sum_{i=1}^{n} \sum_{j=1}^{n} 4p(x_i, x_j)(f(x^{x_i,x_j}) - f(x)) \\
= & \sum_{i=1}^{n} \sum_{j=1}^{n} 2p(x_i, x_j)(f(x^{x_i,x_j}) + f(x^{x_j,x_i}) - 2f(x)) \\
= & \sum_{i=1}^{n} \sum_{j=1}^{n} 2p(x_i, x_j) \\
\times & \sum_{x,y} f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n)(\delta_{x_i,x} - \delta_{x_j,x})(\delta_{x_i,y} - \delta_{x_j,y}) \\
\ge & \ 0
\end{aligned} \tag{3.6}$$

From here on, we can follow the line of thought of proof of Proposition 1.7 in [11]. Since $U_n(t)$ is the semigroup of independent walks, it maps positive definite functions into positive definite functions, we have

$$(\mathcal{L}_n U_n(t) f - \mathcal{L}_{1,n} U_n(t) f) = \mathcal{L}_{2,n} U_n(t) f \ge 0$$

We can then use the variation of constants formula

$$T_n(t)f - U_n(t)f = \int_0^t ds T_n(t-s) \left(\mathcal{L}_{2,n} U_n(s) f\right) \ge 0 \tag{3.7}$$

and remember that $T_n(t)$ is a Markov semigroup and hence maps non-negative functions into non-negative functions. □

4 Stationary measures and self-duality for the SIP

In [8] we found invariant product measures that are reversible for the SIP and we established the self-duality property of the process. Here we recall those results because we will need them to deduce correlation inequalities from the comparison inequality above. Moreover we find the relation between the invariant product measures and the duality functions and from this we deduce ergodicity of these measures.

For a parameter $0 \le \lambda < 1$ we define the probability measure ν_λ on $\mathbb{N}$ via

$$\nu_\lambda(k) = \frac{1}{Z_\lambda}\frac{(2k-1)!!}{2^k k!}\lambda^k, \qquad k \in \mathbb{N} \tag{4.1}$$

with

$$(2k-1)!! = \prod_{j=1}^{k}(2j-1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{2k}e^{-x^2/2}dx, \tag{4.2}$$

$$Z_\lambda = \frac{1}{\sqrt{(1-\lambda)}}$$

and where we make the convention $(-1)!! := 1$. With a slight abuse of notation, we use the same symbol ν_λ to denote the homogeneous product measure on $\Omega = \mathbb{N}^S$ with marginals given by (4.1). The SIP process with generator (2.1) has ν_λ as reversible measure and is self-dual with duality functions defined as follows: for $k \le n$ (with $k, n \in \mathbb{N}$)

$$D(k,n) = \frac{n!}{(n-k)!}\frac{2^k}{(2k-1)!!} \tag{4.3}$$

and $D(k,n) = 0$ for $k > n$. We use the same symbol D for the multivariate version of (4.3), i.e., for $\xi \in \Omega$, finite particle configuration, $(|\xi| = \sum_x \xi_x < \infty)$,

$$D(\xi,\eta) = \prod_{x\in S} D(\xi_x,\eta_x)$$

The SIP is self-dual, with these duality functions, i.e.,

$$\mathbb{E}^{SIP}_\eta D(\xi,\eta_t) = \mathbb{E}^{SIP}_\xi D(\xi_t,\eta) \tag{4.4}$$

where $E^{SIP}_\eta(\cdot)$ denotes expectation in the SIP process η_t, $t \ge 0$, started from η at time $t = 0$. The relation between the polynomials $D(\xi,\eta)$ and the reversible measure ν_λ is easily obtained. Using (4.2) we have

$$\int D(\xi,\eta)\nu_\lambda(d\eta) = \rho_\lambda^{|\xi|} \tag{4.5}$$

where

$$\rho_\lambda = \frac{\lambda}{1-\lambda} \tag{4.6}$$

.

Notice that $D(\delta_x,\eta) = 2\eta_x$ (where δ_x denotes the particle configuration with a single particle at site x), so ρ_λ equals twice the expected number of particles.

From (4.5) and self-duality, we see that the invariance of the measure ν_λ corresponds to the conservation of particles in the dual process, i.e.,

$$\begin{aligned}\int \mathbb{E}^{SIP}_\eta\left(D(\xi,\eta_t)\right)\nu_\lambda(d\eta) &= \mathbb{E}^{SIP}_\xi \int D(\xi_t,\eta)\nu_\lambda(d\eta)\\ &= \mathbb{E}^{SIP}_\xi \rho_\lambda^{|\xi_t|}\\ &= \rho_\lambda^{|\xi|}\\ &= \int \left(D(\xi,\eta)\right)\nu_\lambda(d\eta)\end{aligned} \tag{4.7}$$

From the relation above we can further infer the extremal invariance of the measure ν_λ under the assumption (A2) on the transition probability kernel p. To see this, we denote for two finite particle configurations $\xi \perp \xi'$, if their supports are disjoint, i.e., there are no site $x \in S$ where there are ξ and ξ' particles. If $\xi \perp \xi'$ then $D(\xi + \xi', \eta) = D(\xi, \eta) D(\xi', \eta)$. Since at large $t > 0$, assumption (A2) implies that, in the SIP started with a finite number of particles, particles are with probability close to one at different locations (see Lemma 1 for a proof of this), we have that for ξ' a fixed configuration, the event $\xi_t \perp \xi'$ has probability close to one as $t \to \infty$. Therefore

$$\begin{aligned}
\lim_{t\to\infty} \int \mathbb{E}_\eta^{SIP}\left(D(\xi, \eta_t)\right) D(\xi', \eta)\nu_\lambda(d\eta) &= \lim_{t\to\infty} \mathbb{E}_\xi^{SIP} \int D(\xi_t, \eta) D(\xi', \eta)\nu_\lambda(d\eta) \\
&= \lim_{t\to\infty} \mathbb{E}_\xi^{SIP} \int D(\xi_t, \eta) D(\xi', \eta) I(\xi_t \perp \xi')\nu_\lambda(d\eta) \\
&= \lim_{t\to\infty} \rho_\lambda^{|\xi_t| + |\xi'|} \\
&= \rho_\lambda^{|\xi| + |\xi'|} \\
&= \int D(\xi, \eta)\nu_\lambda(d\eta) \int D(\xi', \eta)\nu_\lambda(d\eta) \qquad (4.8)
\end{aligned}$$

which shows that time dependent correlations of (linear combinations of) $D(\xi, \cdot)$ polynomials decay in the course of time to zero, and hence, by standard argument, ν_λ is mixing and thus ergodic.

5 Correlation inequalities for the SIP

To start with the correlation inequalities that follow from (3.5) in Theorem 1, consider for $\Lambda : S \to [0, 1)$, the inhomogeneous product measure

$$\nu_\Lambda = \otimes_{x\in S} \nu_{\Lambda(x)} \qquad (5.1)$$

where $\nu_{\Lambda(x)}$ is the measure ν_λ of (4.1) with $\lambda = \Lambda(x)$. This inhomogeneous product measure has to be thought of as the analogue of the product of Bernoulli measures in the context of the symmetric exclusion process (SEP). Notice however that in the context of the SEP no distinction can be made between a general product measure and a product of Bernoulli measures, as the single site state space is $\{0, 1\}$. The statement that *negative correlations* are preserved as we evolve the SEP from a product measure, will therefore be replaced here by *positive correlations* are preserved as we evolve the SIP from a measure of type ν_Λ (rather than from a general product measure).

The relation between the inhomogeneous product measure and the duality functions of the SIP reads (using 4.5)

$$\int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \nu_\Lambda(d\eta) = \prod_{i=1}^n \rho(x_i) \qquad (5.2)$$

where $\rho(x) = \Lambda(x)/(1-\Lambda(x))$, and where $\sum_{i=1}^n \delta_{x_i}$ denotes the configuration with particles at positions $(x_1,\ldots,x_n)$. Therefore the map

$$S^n \to \mathbb{R} : (x_1,\ldots,x_n) \mapsto \int D\left(\sum_{i=1}^n \delta_{x_i},\eta\right)\nu_\Lambda(d\eta) = \prod_{i=1}^n \rho(x_i)$$

is clearly positive definite, and we can apply Theorem 1. This gives the following.

Proposition 1. *For all $t \geq 0$, and for all finite particle configurations $\Omega \ni \xi = \sum_{i=1}^n \delta_{x_i}$,*

$$\int \mathbb{E}_\eta\left(D\left(\sum_{i=1}^n \delta_{x_i},\eta_t\right)\right)\nu_\Lambda(d\eta) \geq \prod_{i=1}^n \int \mathbb{E}_\eta\left(D\left(\delta_{x_i},\eta_t\right)\right)\nu_\Lambda(d\eta) \tag{5.4}$$

In particular, when the SIP is started from ν_Λ, the random variables $\{\eta_t(x), x \in S\}$ are positively correlated, i.e., for $(x,y) \in S\times S$

$$\int \mathbb{E}_\eta^{SIP}\left(\eta_x(t)\eta_y(t)\right)\nu_\Lambda(d\eta) \geq \int \mathbb{E}_\eta^{SIP}\left(\eta_x(t)\right)\nu_\Lambda(d\eta)\int \mathbb{E}_\eta^{SIP}\left(\eta_y(t)\right)\nu_\Lambda(d\eta)$$

Proof. Denote by $\mathbb{E}^{SIP}_{x_1,\ldots,x_n}$ expectation in the SIP process started with n particles at positions $(x_1,\ldots,x_n)$, by $\mathbb{E}^{IRW}$ expectation in the process of independent random walkers and $\mathbb{E}^{RW}$ a single random walker expectation. We then have the following chain of inequalities, which is obtained by using sequentially the following: self-duality property (4.4), the comparison inequality (3.5), the relation between the measure ν_Λ and the duality function D (5.2), the independence between random walkers, the fact that a single SIP particle moves as a continuous time random walk, and finally again self-duality (4.4)

$$\begin{aligned}
&\int \mathbb{E}_\eta^{SIP} D\left(\sum_{i=1}^n \delta_{x_i},\eta_t\right)\nu_\Lambda(d\eta)\\
=\;& \mathbb{E}^{SIP}_{x_1,\ldots,x_n}\int D\left(\sum_{i=1}^n \delta_{X_i(t)},\eta\right)\nu_\Lambda(d\eta)\\
\geq\;& \mathbb{E}^{IRW}_{x_1,\ldots,x_n}\int D\left(\sum_{i=1}^n \delta_{X_i(t)},\eta\right)\nu_\Lambda(d\eta)\\
=\;& \mathbb{E}^{IRW}_{x_1,\ldots,x_n}\left(\prod_{i=1}^n \rho(X_i(t))\right)\\
=\;& \prod_{i=1}^n \mathbb{E}^{RW}_{x_i}\rho(X_i(t))\\
=\;& \prod_{i=1}^n \int \mathbb{E}^{SIP}_{x_i}\left(D(\delta_{X_i(t)},\eta)\right)\nu_\Lambda(d\eta)\\
=\;& \prod_{i=1}^n \int \mathbb{E}^{SIP}_{\eta}\left(D(\delta_{x_i},\eta_t)\right)\nu_\Lambda(d\eta)
\end{aligned} \tag{5.5}$$

□

From the analogy with the SEP emphasized above, one could think that (5.4) extends to the case when the SIP process is started from a general product probability measures. However, for general probability measures μ on Ω, the map $\hat{\mu} : S^n \to R$ defined by

$$(x_1, \ldots, x_n) \mapsto \hat{\mu}(x_1, \ldots, x_n) = \int D\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) \mu(d\eta) \tag{5.6}$$

is not necessarily positive definite (as is the case for the special product measures ν_Λ), since we do not have the equality$D\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) = \prod_{i=1}^{n} D(\delta_{x_i}, \eta))$ in general. Notice that this problem does not appear in the context of the SEP, as for that model, the self-duality functions are

$$D_{SEP}\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) = \prod_{i=1}^{n} \eta_{x_i} = \prod_{i=1}^{n} D_{SEP}\left(\delta_{x_i}, \eta\right)$$

and hence automatically the map (5.6) is positive definite in that model.

If however *all* x_i *are different*, we have $D(\sum_{i=1}^{n} \delta_{x_i}, \eta) = \prod_{i=1}^{n} D(\delta_{x_i}, \eta)$, and for every probability measure μ on Ω, the function $\Psi_\mu : S^n \to \mathbb{R}$ defined by

$$\Psi_\mu(x_1, \ldots, x_n) = \int \prod_{i=1}^{n} D(\delta_{x_i}, \eta)\mu(d\eta) \tag{5.7}$$

is clearly positive definite. This, together with the fact that under assumption (A2), a finite number of SIP particles diffuse and therefore eventually will be typically at different positions, suggests that in a stationary measure, the variables η_{x_i} are positively correlated.

To state this result we introduce the class of probability measures with uniform finite moments

$$\mathcal{P}_f =: \{\mu : \forall n \in \mathbb{N},\ \sup_{|\xi|=n} \int D(\xi, \eta)\mu(d\eta) =: M_\mu^n < \infty\} \tag{5.8}$$

For a sequence of measures $\mu_n \in \mathcal{P}_f$, and $\mu \in \mathcal{P}_f$, we define that $\mu_n \to \mu$ if for all ξ finite particle configuration,

$$\lim_{n\to\infty} \int D(\xi, \eta)\mu_n(d\eta) = \int D(\xi, \eta)\mu(d\eta)$$

We can then formulate our next result.

Proposition 2. *Assume (A1) and (A2). Let $\nu \in \mathcal{P}_f$ be a product measure. Let $S(t)$ denote the semigroup of the SIP. Suppose that $\mu = \lim_{n\to\infty} \nu S(t_n)$ for a subsequence $t_n \uparrow \infty$. Then we have $\mu \in \mathcal{P}_f$, μ is invariant and*

$$\hat{\mu}(x_1, \ldots, x_n) \geq \prod_{i=1}^{n} \hat{\mu}(x_i) \tag{5.10}$$

Proof. First, by duality we have, referring to the definition of $\mathcal{P}_f$, for all $t > 0$,

$$\int \mathbb{E}^{SIP}_{\eta} D(\xi, \eta_t)\nu(d\eta) = \mathbb{E}^{SIP}_{\xi} \int D(\xi_t, \eta)\nu(d\eta) \leq M_{\nu}^{|\xi|} < \infty$$

which shows that both $\nu S(t_n)$ and μ are elements of $\mathcal{P}_f$. The invariance of μ follows from duality, $\nu \in \mathcal{P}_f$ and Lemma 1.26 in [11], chapter V.

To proceed with the proof of the proposition, we start with the following lemma, which ensures that, under condition (A2), any number of SIP particles will eventually be at different locations.

Lemma 1. *Assume (A2). Start the finite SIP with particles at locations* $\{x_1, \ldots, x_n\}$, *then*

$$\lim_{t\to\infty} \mathbb{P}^{SIP}_{x_1,\ldots,x_n} (\exists i \neq j : X_i(t) = X_j(t)) = 0 \tag{5.12}$$

Proof. Put $\eta := \sum_{i=1}^n \delta_{x_i}$. Using self-duality we can write

$$\begin{aligned}
\mathbb{P}^{SIP}_{\eta} (\exists i \neq j : X_i(t) = X_j(t)) &\leq \sum_z \mathbb{P}^{SIP}_{\eta} \big(\eta_t^2(z) - \eta_t(z) > 1\big) \\
&\leq \sum_z \mathbb{E}^{SIP}_{\eta} (\eta_t^2(z) - \eta_t(z)) \\
&= \frac{3}{4} \sum_z \mathbb{E}^{SIP}_{\eta} (D(2\delta_z, \eta_t)) \\
&= \frac{3}{4} \sum_z \mathbb{E}^{SIP}_{z,z} (D(\delta_{X_t} + \delta_{Y_t}, \eta)) \\
&\leq 3 \sum_z \mathbb{E}^{SIP}_{z,z} (\eta(X_t)\eta(Y_t)) \\
&= 3 \sum_z \sum_{i,j=1}^n \mathbb{E}^{SIP}_{z,z} (I(X_t = x_i) I(Y_t = x_j)) \\
&\leq 3n^2 \sup_{x,y} \mathbb{P}^{SIP}_{x,y} (X_t = Y_t) \qquad (5.13)
\end{aligned}$$

where in the last step we used the symmetry of the transition probabilities of the SIP (with two particles). □

We now proceed with the proof of the proposition. For $x_1, \ldots, x_n \in S$ we define

$$\left| D(\sum_{i=1}^n \delta_{x_i}, \eta) - \prod_{i=1}^n D(\delta_{x_i}, \eta) \right| = \Delta(x_1, \ldots, x_n, \eta) \tag{5.14}$$

We have that $\Delta(x_1, \ldots, x_n, \eta) = 0$ if all x_i are different, i.e., if $|\{x_1, \ldots, x_n\}| = n$. Since by assumption (A2) and Lemma 1, the probability that two SIP walkers out of a finite number n of them occupy the same position, i.e. $X_i(t) = X_j(t)$ for some $i \neq j$, vanishes in the limit $t \to \infty$, we conclude, using $\nu \in \mathcal{P}_f$, for any $x_1, \ldots, x_n \in S$,

$$\lim_{t\to\infty} \int \mathbb{E}^{SIP}_{x_1,\ldots,x_n} \Delta(X_1(t), \ldots, X_n(t), \eta)\nu(d\eta) = 0 \tag{5.15}$$

Moreover from the comparison inequality (3.5) we have, using the notation (5.7)

$$\begin{aligned}
\mathbb{E}^{SIP}_{x_1,\ldots,x_n}\Psi_\nu(X_1(t),\ldots,X_n(t)) &\geq \mathbb{E}^{IRW}_{x_1,\ldots,x_n}\Psi_\nu(X_1(t),\ldots,X_n(t)) \\
&= \mathbb{E}^{IRW}_{x_1,\ldots,x_n}\int\prod_{i=1}^n D\left(\delta_{X_i(t)},\eta\right)\nu(d\eta) \\
&= \prod_{i=1}^n \mathbb{E}^{RW}_{x_i}\int D(\delta_{X_i(t)},\eta)\nu(d\eta)+\epsilon(t) \qquad (5.16)
\end{aligned}$$

where $\epsilon(t)\to 0$ as $t\to\infty$ by assumption (A1), i.e., for large $t>0$, independent random walkers are at different locations with probability close to one. Therefore, using the definition (5.6), the self-duality property (4.4), the equation (5.15), the equation (5.16), and taking limits along the subsequence t_n we have

$$\begin{aligned}
\hat\mu(x_1,\ldots,x_n) &= \lim_{t\to\infty}\int\mathbb{E}^{SIP}_\eta D\left(\sum_{i=1}^n\delta_{x_i},\eta_t\right)\nu(d\eta) \\
&= \lim_{t\to\infty}\int\mathbb{E}^{SIP}_{x_1,\ldots,x_n} D\left(\sum_{i=1}^n\delta_{X_i(t)},\eta\right)\nu(d\eta) \\
&= \lim_{t\to\infty}\mathbb{E}^{SIP}_{x_1,\ldots,x_n}\Psi_\nu(X_1(t),\ldots,X_n(t)) \\
&\geq \lim_{t\to\infty}\prod_{i=1}^n\mathbb{E}^{RW}_{x_i}\int D(\delta_{X_i(t)},\eta)\nu(d\eta) \\
&= \prod_{i=1}^n\hat\mu(x_i) \qquad (5.17)
\end{aligned}$$

□

6 Correlation inequalities for the Brownian momentum process

The Brownian momentum process is a system of interacting diffusions, initially introduced as a model of heat conduction in [6], and analyzed via duality in [7]. It is defined as a Markov process on $X=\mathbb{R}^S$ via the formal generator on local functions:

$$L_{BMP}f(\eta)=\left(\sum_{x,y\in S}p(x,y)\left(\eta_x\frac{\partial}{\partial\eta_y}-\eta_x\frac{\partial}{\partial\eta_y}\right)^2\right)f(\eta) \qquad (6.1)$$

The η_x have to be thought of as momenta of an "oscillator" associated to site $x\in S$. The local kinetic energy η_x^2 has to be thought of as the analogue of the number of particles at site x in the SIP. The expectation of η_x^2 is interpreted as the local temperature at x.

Defining the polynomials

$$D(n,z) = \frac{z^{2n}}{(2n-1)!!}$$

we have the duality function $D(\xi,\cdot)$ defined on X and indexed by finite particle configurations $\xi \in \mathbb{N}^S$, $\sum_x \xi_x < \infty$:

$$D(\xi,\eta) = \prod_{x\in S} D(\xi_x,\eta_x) \tag{6.2}$$

In [7], [8], we proved the duality relation

$$\mathbb{E}_\eta^{BMP}\left(D(\xi,\eta_t)\right) = \mathbb{E}_\xi^{SIP}\left(D(\xi_t,\eta)\right) \tag{6.3}$$

As before, for $x_1,\ldots,x_n \in S$ we denote by $\sum_{i=1}^n \delta_{x_i}$ the particle configuration obtained by putting a particle at each x_i.

Let μ be a product of Gaussian measures on X, with site-dependent variance, i.e., for a function $\rho : S \to [0,\infty)$, we define

$$\mu_\rho = \otimes_{x\in S} \nu_{\rho(x)}(d\eta_x) \tag{6.4}$$

where

$$\nu_{\rho(x)}(d\eta_x) = \frac{e^{-\eta_x^2/2\rho(x)}}{\sqrt{2\pi\rho(x)}} d\eta_x$$

is the Gaussian measure on $\mathbb{R}$ with mean zero and variance $\rho(x)$. Then we have

$$\int D(\sum_{i=1}^n \delta_{x_i},\eta)\mu_\rho(d\eta) = \prod_{i=1}^n \rho(x_i) \tag{6.5}$$

From this expression, it is obvious that the map

$$S^n \to \mathbb{R} : (x_1,\ldots,x_n) \mapsto \int D(\sum_{i=1}^n \delta_{x_i},\eta)\mu_\rho(d\eta) \tag{6.6}$$

is positive definite. Therefore, combining the duality property between BMP process and SIP process, (6.3), with Theorem 1 we have the inequality

$$\begin{aligned}
& \int \mathbb{E}^{BMP}_{\eta} D(\sum_{i=1}^{n} \delta_{x_i}, \eta_t)\mu_\rho(d\eta) \\
= \; & \mathbb{E}^{SIP}_{x_1,\ldots,x_n} \int D(\sum_{i=1}^{n} \delta_{X_i(t)}, \eta)\mu_\rho(d\eta) \\
\geq \; & \mathbb{E}^{IRW}_{x_1,\ldots,x_n} \int D(\sum_{i=1}^{n} \delta_{X_i(t)}, \eta)\mu_\rho(d\eta) \\
= \; & \mathbb{E}^{IRW}_{x_1,\ldots,x_n} \left(\prod_{i=1}^{n} \int D(\delta_{X_i(t)}, \eta)\mu_\rho(d\eta) \right) \\
= \; & \mathbb{E}^{IRW}_{x_1,\ldots,x_n} \left(\prod_{i=1}^{n} \rho(X_i(t)) \right) \\
= \; & \prod_{i=1}^{n} \mathbb{E}^{RW}_{x_i} \rho(X_i(t)) \\
= \; & \prod_{i=1}^{n} \int \mathbb{E}^{SIP}_{x_i} (D(\delta_{X_i(t)}, \eta))\mu_\rho(d\eta) \\
= \; & \prod_{i=1}^{n} \int \mathbb{E}^{BMP}_{\eta} (D(\delta_{x_i}, \eta_t))\mu_\rho(d\eta) \qquad (6.7)
\end{aligned}$$

which is the analogue of Proposition 1 for the BMP process.

In words, it means that the "non-equilibrium temperature profile" is above the temperature profile predicted from the discrete diffusion equation. It also implies that the variables $\{\eta_x^2 : x \in S\}$ are positively correlated under the measure $(\mu_\rho)_t$ for all choices of ρ, $t > 0$.

More precisely, if we denote

$$\rho_t(x) = \mathbb{E}_x \rho(X_t)$$

then we have that η_x^2 at time t has expectation $\rho_t(x)$ when the starting measure is μ_ρ (since a single particle in the SIP moves as a continuous time random walk). If we denote by μ_{ρ_t} the Gaussian product measure with mean zero and variance $\mu_{\rho_t(x)}(\eta_x^2) = \rho_t(x)$, then the measure $(\mu_\rho)_t$ (evolved for a time t under the BMP process) dominates the measure μ_{ρ_t}, i.e., for all $\xi \in \mathbb{N}^S$ finite particle configuration, we have

$$\int D(\xi, \eta)(\mu_\rho)_t(d\eta) \geq \int D(\xi, \eta)(\mu_{\rho_t})(d\eta) \qquad (6.8)$$

Similarly, we obtain an analogous correlation inequality for the BMP for measure obtained as a limit of product measures. We define

$$\mathcal{P}_f(X) = \{\mu : \forall n \in \mathbb{N} : \sup_{|\xi|=n} \int D(\xi, \eta)\mu(d\eta) < \infty\}$$

Proposition 3. *Assume (A1) and (A2). Suppose $\nu \in \mathcal{P}_f(X)$ is a product measure and μ is a limit point of the set $\{\nu S(t) : t \geq 0\}$, where $S(t)$ denotes the semigroup of the BMP process. Then we have the inequality*

$$\hat{\mu}(x_1, \ldots, x_n) \geq \prod_{i=1}^{n} \hat{\mu}(x_i)$$

7 Generalization to the SIP(m) processes

The $SIP(m)$ process is defined as the process on $\Omega = \mathbb{N}^{\mathbb{Z}^d}$ with generator defined on the core of local functions by

$$Lf(\eta) = \sum_{x,y \in S} p(x,y) 2\eta_x (m + 2\eta_y) \left(f(\eta^{x,y}) - f(\eta) \right) \tag{7.1}$$

The SIP process is the case $m = 1$.

This model has reversible product measures with marginals

$$\nu_\lambda^m(n) = \frac{1}{Z_{\lambda,m}} \frac{\lambda^n}{n!} \frac{\Gamma(\frac{m}{2} + n)}{\Gamma(\frac{m}{2})}, \qquad n \in \mathbb{N} \tag{7.2}$$

where $0 \leq \lambda < 1$ is a parameter, $\Gamma(r)$ denotes the gamma-function, and where the normalizing constant

$$Z_{\lambda,m} = \left(\frac{1}{1 - \lambda} \right)^{m/2}$$

This can be seen immediately by verifying the detailed balance condition.

Notice that for $m = 2$, ν_λ^m is a geometric distribution (starting from zero), i.e., $\nu_\lambda^2(n) = \lambda^n (1 - \lambda), n \in \mathbb{N}$. Moreover, the measures ν^m have the following convolution property

$$\nu_\lambda^m * \nu_\lambda^l = \nu_\lambda^{m+l} \tag{7.3}$$

where $*$ denotes convolution, i.e., a sample from $\nu_\lambda^m * \nu_\lambda^l$ is obtained by site-wise addition of a sample from ν_λ^m and an independent sample from ν_λ^l.

The $SIP(m)$ process is self-dual with duality functions now given by $D_m(\xi, \eta) = \prod_x D_m(\xi_x, \eta_x)$, with

$$D_m(k, n) = \frac{n!}{(n-k)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} + k\right)} \tag{7.4}$$

The relation between the polynomials D_m and the measure ν_λ^m reads

$$\int D_m(\xi, \eta) \nu_\lambda^m(d\eta) = \left(\frac{\lambda}{1 - \lambda} \right)^{|\xi|} \tag{7.5}$$

as follows from a simple computation using $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$.

So we see once more that the invariance of the measures ν_λ^m corresponds to conservation of total particle number in the dual process. In the same way as for the SIP, it also implies that the measures ν_λ^m are extremal invariant, and analog Propositions to Propositions 1 and 2 can be proved in the same way as in the $m = 1$ case.

Proposition 4. *Denote, for $\Lambda : S \to [0,1)$ the product measure*

$$\nu_\Lambda^m = \otimes_x \nu_{\Lambda(x)}^m \tag{7.7}$$

where $\nu_{\Lambda(x)}$ is the measure ν_λ of (7.2) with $\lambda = \Lambda(x)$. Then we have the inequality

$$\int \mathbb{E}_\eta^{SIP(m)} \left(D_m \left(\sum_{i=1}^n \delta_{x_i}, \eta_t \right) \right) \nu_\Lambda^m(d\eta) \geq \prod_{i=1}^n \int \mathbb{E}_\eta^{SIP(m)} \left(D_m \left(\delta_{x_i}, \eta_t \right) \right) \nu_\Lambda^m(d\eta)) \tag{7.8}$$

Proposition 5. *Assume (A1) and (A2). Let $\nu \in \mathcal{P}_f$ be a product measure on Ω, suppose that $\nu S(t)$ has a limit point along a sequence of times $t_n \uparrow \infty$, then for the limit point μ, we have the correlation inequality*

$$\hat{\mu}^m(x_1, \ldots, x_n) \geq \prod_{i=1}^n \hat{\mu}^m(x_i)$$

where

$$\hat{\mu}^m(x_1, \ldots, x_n) = \int D_m \left(\sum_{i=1}^n \delta_{x_i}, \eta \right) \mu(d\eta)$$

Finally we explain how the convolution property (7.3) arises from an "additivity" property at the process level. Consider the following SIP on a graph of the form $G = S \times \{1, 2, \ldots, k\}$. Vertices in G are denoted (i, α). The SIP which we consider on G has generator

$$Lf(\eta) = \sum_{i,j \in S} \sum_{\alpha, \beta \in \{1,2,\ldots,k\}} p(i,j) 2\eta(i,\alpha)(m_\beta + 2\eta(j,\beta))(f(\eta^{(i,\alpha),(j,\beta)}) - f(\eta)) \tag{7.10}$$

This is interpreted as follows: every site in S has k levels and SIP-particles jump with an underlying random walk kernel that does not depend on the level.

Consider then the reduced configuration $h(\eta)_i = \sum_{\alpha=1}^k \eta(i,\alpha)$ $(i \in S)$, and a function of the form $f = \psi \circ h(\eta)$. The generator applied to this type of function yields

$$Lf(\eta) = \sum_{i,j \in S} p(i,j) 2h(\eta)_i \left(\sum_{\beta=1}^k m_\beta + 2h(\eta)_j \right) \left(\psi(h(\eta)^{(i,j)}) - \psi(\eta) \right) \tag{7.11}$$

Therefore the process $h(\eta_t)$ is again a Markov process which is exactly the SIP (with particle configurations on S) with underlying kernel $p(i,j)$ and $m = \sum_{\beta=1}^k m_\beta$.

8 Asymmetric generalization

For simplicity we first illustrate here the one-dimensional nearest neighbor case. The asymmetric modification of the SIP is then the process with generator

$$\begin{aligned} L_{p,q}^{ASIP} f(\eta) &= \sum_{i \in \mathbb{Z}} 2p\eta_i(1 + 2\eta_{i+1})(f(\eta^{i,i+1}) - f(\eta)) \\ &+ \sum_{i \in \mathbb{Z}} 2q\eta_{i+1}(1 + 2\eta_i)(f(\eta^{i+1,i}) - f(\eta)) \end{aligned} \tag{8.1}$$

where $1 > p > 1/2$, $q = (1-p)$.

The following proposition shows that the measures ν_λ are still invariant for this asymmetric modification.

Proposition 6. *Let ν_λ be the product measure with marginals defined as in* (4.1). *Then for every $1 > p > 1/2$, ν_λ is a stationary measure for the ASIP with generator* (8.1)

Proof. Using detailed balance of the measure $\nu = \nu_\lambda$ for the SIP, we have

$$\nu(\eta)2\eta_i(1+2\eta_{i+1}) = \nu(\eta^{i,i+1})2\eta_{i+1}^{i,i+1}(2\eta_i^{i,i+1}+1)$$

and

$$\nu(\eta)2\eta_{i+1}(1+2\eta_i) = \nu(\eta^{i+1,i})2\eta^{i+1,i}(2\eta_{i+1}^{i+1,i}+1)$$

Using this one easily computes for $f : \Omega \to \mathbb{R}$ a local function,

$$\begin{aligned}\int L_{p,q}^{ASIP} f(\eta)d\nu &= \lim_{N\to\infty}\sum_{i=-N}^{N}\int 2p\eta_i(1+2\eta_{i+1})(f(\eta^{i,i+1})-f(\eta))d\nu \\ &+ \lim_{N\to\infty}\sum_{i=-N}^{N}\int 2q\eta_{i+1}(1+2\eta_i)(f(\eta^{i+1,i})-f(\eta))d\nu \\ &= 2(p-q)\lim_{N\to\infty}\int\sum_{i=-N}^{N}(\eta_{i+1}-\eta_i)f(\eta)d\nu = 0 \qquad (8.3)\end{aligned}$$

□

This proposition can now be generalized easily to the case $S = \mathbb{Z}^d$ and translation invariant underlying random walk. I.e., for $p : \mathbb{Z}^d \to [0,1]$ with $\sum_x p(x) = 1, \sum_x |x|p(x) < \infty$ consider the generator

$$L_p^{ASIP} f(\eta) = \sum_{i,j\in\mathbb{Z}^d} 2p(j-i)\eta_i(1+2\eta_j)(f(\eta^{i,j})-f(\eta)) \qquad (8.4)$$

then we have

Proposition 7. *The product measure ν_λ with marginals given by* (4.1), *is stationary for the process with generator L_p^{ASIP}*

Proof. Using detailed balance for $\nu = \nu_\lambda$ (for the SIP), we have

$$\nu(\eta)2\eta_j(1+2\eta_i) = \nu(\eta^{j,i})2\eta_i^{j,i}(1+2\eta_j^{j,i})$$

which gives, for $f : \Omega \to \mathbb{R}$ a local function,

$$\int L_p^{ASIP} f(\eta)d\nu = \lim_{N\to\infty}\sum_{i,j\in\mathbb{Z}^d,|i-j|\le N} p(i-j)\int 2(\eta_j-\eta_i)fd\nu = 0 \qquad (8.6)$$

□

An asymmetric version of the $SIP(m)$ with the same product measures ν_λ^m as invariant measures is given by the generator

$$L_{\pi,m}^{ASIP} f(\eta) = \sum_{x,y\in S} \pi(y-x) 2\eta_x (m+2\eta_y) \left(f(\eta^{x,y}) - f(\eta)\right)$$

where $\pi(x) \geq 0$, $\sum_x \pi(x) = 1, \sum_x |x|\pi(x) < \infty$.

8.1 Inhomogeneous invariant measures

In this section we compute reversible infinite measures for the asymmetric $SIP(m)$. Because of the attractive interaction between the particles, it is intuitively clear that a profile where the expected number of particles at site i increases as $i \to \infty$ cannot persist in time (particles would run of to plus infinity). Nevertheless, we show that there exist non-translation invariant *infinite measures* which are reversible for the process. This phenomenon of having both translation invariant non-reversible probability measures and non-translation invariant reversible infinite measures is related to both the unbounded state space and the attractive interaction between the particles.

Consider the nearest neighbor asymmetric $ASIP(m)$, with generator

$$\begin{aligned} L_{p,q,m}^{ASIP} f(\eta) &= \sum_{i\in\mathbb{Z}} 2p\eta_i(m+2\eta_{i+1})(f(\eta^{i,i+1}) - f(\eta)) \\ &+ \sum_{i\in\mathbb{Z}} 2q\eta_{i+1}(m+2\eta_i)(f(\eta^{i+1,i}) - f(\eta)) \end{aligned} \tag{8.7}$$

We suppose $p > q$, i.e., particles drift to the right. We look for non-translation invariant product measures that are reversible under this process. I.e., we look for a measure

$$\nu(d\eta) = \otimes_{i\in\mathbb{Z}} \nu_i(d\eta_i)$$

which satisfies detailed balance. The detailed balance condition gives the recursion

$$\left(\frac{\nu_i(n)}{\nu_i(n-1)} \frac{2n}{m+2n-2}\right) \left(\frac{\nu_{i+1}(k+1)}{\nu_{i+1}(k)} \frac{2(k+1)}{m+2k}\right)^{-1} = \frac{q}{p} \tag{8.8}$$

To solve this recursion, we make the Ansatz

$$\left(\frac{\nu_i(n)}{\nu_i(n-1)} \frac{2n}{m+2n-2}\right) = z^i \alpha(n)$$

which gives $z = p/q$ and

$$\begin{aligned} \nu_i(n) &= \nu_i(0) \left(\frac{p}{q}\right)^{ni} \lambda^n \prod_{k=1}^{n} \frac{m+2k-2}{2k} \\ &= \nu_i(0) \left(\frac{p}{q}\right)^{ni} \lambda^n \frac{\Gamma\left(\frac{m}{2}+n\right)}{\Gamma\left(\frac{m}{2}\right) n!} \end{aligned} \tag{8.9}$$

The series

$$\sum_{n=0}^{\infty} \left(\frac{p}{q}\right)^{ni} \lambda^n \frac{\Gamma\left(\frac{m}{2}+n\right)}{\Gamma\left(\frac{m}{2}\right) n!}$$

converges for

$$\left(\frac{p}{q}\right)^{i} \lambda < 1$$

and diverges otherwise. Therefore, the measure ν_i on $\mathbb{N}$ is a finite measure for $i < (\log(1/\lambda))(\log(p/q))^{-1}$, and infinite for $i \geq (\log(1/\lambda))(\log(p/q))^{-1}$.

For the sites $i < (\log(1/\lambda))(\log(p/q))^{-1}$, the expected number of particles is equal to

$$\mathbb{E}(\eta_i) = \frac{m}{2} \frac{\left(\frac{p}{q}\right)^{i} \lambda}{1 - \left(\frac{p}{q}\right)^{i} \lambda}$$

which diverges in the "limit" $i \to (\log(1/\lambda))(\log(p/q))^{-1}$.

9 The boundary driven $SIP(m)$

In this section we consider the non-equilibrium one-dimensional model that is obtained by considering particle reservoirs attached to the first and last particle of the chain. We will show that, if one requires reversibility w.r.t. the measure ν_λ^m *and* duality with absorbing boundaries, this uniquely fixes the birth and death rates at the boundaries.

The generator of the boundary driven $SIP(m)$ on a chain $\{1, \ldots, N\}$ driven at the end points, reads

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_N + \mathcal{L}_{bulk} \tag{9.1}$$

where $\mathcal{L}_{bulk}$ denotes the $SIP(m)$ generator, with nearest neighbor random walk as underlying kernel, i.e.,

$$\mathcal{L}_{bulk} f(\eta) = \sum_{x \in \{1,\ldots,N-1\}} 2\eta_x(m+2\eta_{x+1}) \left(f(\eta^{x,x+1}) - f(\eta)\right) + 2\eta_{x+1}(m+2\eta_x) \left(f(\eta^{x+1,x}) - f(\eta)\right) \tag{9.2}$$

and where $\mathcal{L}_1, \mathcal{L}_N$ are birth and death processes on the first, resp. N-th variable, i.e.,

$$\mathcal{L}_1 f(\eta) = d_L(\eta_1)(f(\eta - \delta_1) - f(\eta)) + b_L(\eta_1)(f(\eta + \delta_1) - f(\eta))$$

and

$$\mathcal{L}_N f(\eta) = d_R(\eta_N)(f(\eta - \delta_N) - f(\eta)) + b_R(\eta_N)(f(\eta + \delta_N) - f(\eta))$$

These generators model contact with the left, resp. right particle reservoir.

The rates d_L, b_L, d_R, b_R are chosen such that detailed balance is satisfied w.r.t. the measure ν_λ^m, with $\lambda = \lambda_L$ for d_L, b_L, and $\lambda = \lambda_R$ for d_R, b_R. More precisely, this means that these rates satisfy

$$b_\alpha(k)\nu_{\lambda_\alpha}^m(k) = d_\alpha(k+1)\nu_{\lambda_\alpha}^m(k+1) \tag{9.3}$$

for $\alpha \in \{L, R\}$.

To state our duality result, we consider functions $\mathcal{D}(\xi, \eta)$ indexed by particle configurations ξ on $\{0, \ldots, N+1\}$ defined by

$$\mathcal{D}(\xi, \eta) = \rho_L^{|\xi_0|} D(\xi_{\{1,\ldots,N\}}, \eta) \rho_R^{|\xi_{N+1}|} \tag{9.4}$$

where $\rho_\alpha = \rho_{\lambda_\alpha} = \lambda_\alpha/(1-\lambda_\alpha)$, and where we remember that

$$D(k, n) = \frac{n!}{(n-k)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+k\right)}$$

is the duality function for the $SIP(m)$. I.e., for the "normal" sites $\{1, \ldots, N\}$ we simply have the old duality functions, and for the "added" sites $\{0, N+1\}$ we have the expectation of the duality function over the measure ν_λ^m.

We now want duality to hold with duality functions $\mathcal{D}$, and with a dual process that behaves in the bulk as the $SIP(m)$, and which has *absorbing boundaries* at $\{0, N+1\}$. More precisely, we want the generator of the dual process to be

$$\hat{\mathcal{L}} = \mathcal{L}_{bulk} + \hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_N \tag{9.5}$$

with $\mathcal{L}_{bulk}$ given by (9.2), and

$$\hat{\mathcal{L}}_1 f(\xi) = \xi_1 \left(f(\xi^{1,0}) - f(\xi)\right)$$

$$\hat{\mathcal{L}}_N f(\xi) = \xi_N \left(f(\xi^{N,N+1}) - f(\xi)\right)$$

for $\xi \in \mathbb{N}^{\{0,1\ldots,N+1\}}$. The duality relation then reads, as usual,

$$\left(\mathcal{L}\mathcal{D}(\xi, \cdot)\right)(\eta) = \left(\hat{\mathcal{L}}\mathcal{D}(\cdot, \eta)\right)(\xi) \tag{9.6}$$

Since self-duality is satisfied for the bulk generator with the choice (9.4), i.e., since

$$\left(\mathcal{L}_{bulk}\mathcal{D}(\xi, \cdot)\right)(\eta) = \left(\mathcal{L}_{bulk}\mathcal{D}(\cdot, \eta)\right)(\xi)$$

(9.6) will be satisfied if we have the following relations at the boundaries: for all $k \le n$:

$$\begin{aligned} & b_\alpha(n)(D(k, n+1) - D(k, n)) + d_\alpha(n)(D(k, n-1) - D(k, n)) \\ = \; & k(D(k-1, n)\rho_\alpha - D(k, n)) \end{aligned} \tag{9.7}$$

where $\alpha \in \{L, R\}$.

From detailed balance (9.3) we obtain

$$d_\alpha(n) = \frac{1}{\lambda_\alpha} \left(\frac{n}{\frac{m}{2}+n-1}\right) b_\alpha(n-1) \tag{9.8}$$

Working out (9.7) gives, using (7.4),

$$
\begin{aligned}
& b_\alpha(n)\left(\frac{n+1}{n+1-k}-1\right)+d_\alpha(n)\left(\frac{n-k}{n}-1\right) \\
= \quad & k\left(\frac{\left(\frac{m}{2}+k-1\right)\rho_\alpha}{n-k+1}-1\right) \qquad (9.9)
\end{aligned}
$$

which simplifies to

$$
\frac{b_\alpha(n)}{n+1-k}-\frac{d_\alpha(n)}{n}=\left(\frac{\left(\frac{m}{2}+k-1\right)\rho_\alpha}{n-k+1}-1\right) \qquad (9.10)
$$

Choosing

$$
d_\alpha(n)=\frac{n}{1-\lambda_\alpha} \qquad (9.11)
$$

and by the detailed balance condition (9.8),

$$
b_\alpha(n)=\left(\frac{m}{2}+n\right)\frac{\lambda_\alpha}{1-\lambda_\alpha} \qquad (9.12)
$$

it is then an easy computation to see that (9.7) is satisfied with the choices (9.11), (9.12). Indeed, (9.10) reduces to the simple identity

$$
\left(\frac{m}{2}+n\right)\left(\frac{\lambda}{1-\lambda}\right)\frac{1}{n+1-k}-\frac{1}{1-\lambda}=\frac{\frac{m}{2}+k-1}{n+1-k}\left(\frac{\lambda}{1-\lambda}\right)-1
$$

We remark that the requirement of detailed balance alone is not sufficient to fix the rates uniquely. However, the additional duality constraint (9.7) does fix the rates to the unique expression given by (9.11) and (9.12).

As a consequence of duality with duality functions (9.4), we have that the boundary driven $SIP(m)$ with generator (9.1) has a unique stationary measure $\mu_{L,R}$ for which expectations of the polynomials $D(\xi,\eta)$ are given in terms of absorption probabilities:

$$
\begin{aligned}
\int D(\xi,\eta)\mu_{L,R}(d\eta) &= \lim_{t\to\infty}\int \mathbb{E}_\eta\mathcal{D}(\xi,\eta_t) \\
&= \lim_{t\to\infty}\int \hat{\mathbb{E}}_\xi\mathcal{D}(\xi_t,\eta) \\
&= \sum_{k,l:k+l=|\xi|} \rho_L^k\rho_R^l\hat{\mathbb{P}}_\xi\left(\xi_\infty=k\delta_0+l\delta_{N+1}\right) \qquad (9.13)
\end{aligned}
$$

Here, $\hat{\mathbb{E}}_\xi$ denotes expectation in the dual process (which is absorbing at $\{0,N+1\}$) starting from ξ. In particular, since a single $SIP(m)$ particle performs continuous time simple random walk, we have a linear density profile, i.e.,

$$
\int D(\delta_i,\eta)\mu_{L,R}(d\eta)=\rho_L\left(1-\frac{i}{N+1}\right)+\rho_R\frac{i}{N+1} \qquad (9.14)
$$

9.1 Correlation inequality for the boundary driven $SIP(m)$

For $x_1, \ldots, x_n \in \{1, \ldots, N\}$ let us denote by $(X_1(t), \ldots, X_n(t))$ the positions of particles at time t evolving according to the $SIP(m)$ with absorbing states $\{0, N+1\}$, i.e., according to the generator (9.5), and initially at positions $x_1, \ldots, x_n$. Let $(Y_1(t), \ldots, Y_n(t))$ denote the positions at time t of independent random walkers (jumping at rate 2) absorbed (at rate 1) at $\{0, N+1\}$, initially at positions $x_1, \ldots, x_n$. Since the absorption parts of the generators of $(X_1(t), \ldots, X_n(t))$ and $(Y_1(t), \ldots, Y_n(t))$ are the same, we have the same inequality for expectations of positive definite functions as in Theorem 1. Therefore, we have the following result on positivity of correlations in the stationary state. This has once more to be compared to the analogous situation of the boundary driven exclusion process, where the covariances of site-occupations are negative.

Proposition 8. *Let $\mu_{L,R}$ denote the unique stationary measure of the process with generator* (9.1). *Let $x_1, \ldots, x_n \in \{1, \ldots, N\}$, then we have*

$$\int D\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) \mu_{L,R}(d\eta) \geq \prod_{i=1}^{n} \int D(\delta_{x_i}, \eta) \mu_{L,R}(d\eta) \tag{9.16}$$

In particular, $\eta_x, x \in \{1, \ldots, N\}$ are positively correlated under the measure $\mu_{L,R}$.

Proof. Start from the measure ν_λ^m. Define the map $\{0, \ldots, N+1\}^n \to \mathbb{R}$:

$$(x_1, \ldots, x_n) \mapsto \int \mathcal{D}\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) \nu_\lambda^m(d\eta) = \prod_{i=0}^{n} \rho(x_i) \tag{9.17}$$

where $\rho(x) = \frac{\lambda}{1-\lambda}$ for $x \in \{1, \ldots, N\}$ and $\rho(0) = \rho_L, \rho(N+1) = \rho_R$. This is clearly positive definite. Therefore, for $x_1, \ldots, x_n \in \{1, \ldots, N\}$, we have

$$\begin{aligned}
\int D\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) \mu_{L,R}(d\eta) &= \lim_{t\to\infty} \int \mathbb{E}_\eta \mathcal{D}\left(\sum_{i=1}^{n} \delta_{x_i}, \eta_t\right) \nu_\lambda^m(d\eta) \\
&= \lim_{t\to\infty} \int \hat{\mathbb{E}}^{SIP,abs}_{x_1,\ldots,x_n} \left(\mathcal{D}(\sum_{i=1}^{n} \delta_{X_i(t)}, \eta)\right) \nu_\lambda^m(d\eta) \\
&\geq \lim_{t\to\infty} \hat{\mathbb{E}}^{IRW,abs}_{x_1,\ldots,x_n} \left(\int \mathcal{D}(\sum_{i=1}^{n} \delta_{X_i(t)}, \eta) \nu_\lambda^m(d\eta)\right) \\
&= \prod_{i=1}^{n} \lim_{t\to\infty} \hat{\mathbb{E}}^{IRW,abs}_{x_i} \rho(X_i(t)) \\
&= \prod_{i=1}^{n} \int D\left(\delta_{x_i}, \eta\right) \mu_{L,R}(d\eta)
\end{aligned} \tag{9.18}$$

where we denoted $\hat{\mathbb{E}}^{SIP,abs}$ for expectation over $SIP(m)$ particles absorbed at $\{0, N+1\}$, and $\hat{\mathbb{E}}^{IRW,abs}$ for expectation over a system of independent random walkers (jumping at rate 2) absorbed (at rate 1) at $\{0, N+1\}$. □

Remark 1. *Proposition 8 is in agreement with the findings of [7], where the covariance of η_i, η_j in the measure $\mu_{L,R}$ was computed explicitly, and turned out to be positive.*

Remark 2. *For the nearest neighbor SEP on $\{1, \ldots, N\}$ driven at the boundaries, we have self-duality with absorption of dual particles at $\{0, N = 1\}$ and duality function*

$$\mathcal{D}_{SEP}\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) = \prod_{i=1}^{n} \eta_{x_i}$$

where $\eta_0 := \rho_L, \eta_{N+1} = \rho_R$. Since for SEP particles we have the comparison inequality of Liggett, we have as an analogue of (9.16) *in the SEP context,*

$$\int \prod_{i=1}^{n} \eta_{x_i} \ \mu_{L,R}(d\eta) \leq \prod_{i=1}^{n} \int \eta_{x_i} \ \mu_{L,R}(d\eta)$$

i.e., η_{x_i} are negatively correlated. This is in agreement with the results in [13], where the two-point function of the measure $\mu_{L,R}$ is computed, and with the work of [5], where some multiple correlations are explicitly computed.

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