Reviews in Mathematical Physics
Vol. 16, No. 5 (2004) 629–637
© World Scientific Publishing Company

THE THERMODYNAMIC LIMIT FOR FINITE DIMENSIONAL CLASSICAL AND QUANTUM DISORDERED SYSTEMS

PIERLUIGI CONTUCCI* and CRISTIAN GIARDINÀ†

Dipartimento di Matematica, Università di Bologna
40127 Bologna, Italy
**contucci@dm.unibo.it*
†giardina@dm.unibo.it

JOSEPH PULÉ

Department of Mathematical Physics, University College Dublin
Belfield, Dublin 4, Ireland
Joe.Pule@ucd.ie

Received 27 January 2004
Revised 15 April 2004

We provide a very simple proof for the existence of the thermodynamic limit for the quenched specific pressure for classical and quantum disordered systems on a d-dimensional lattice, including spin glasses. We develop a method which relies simply on Jensen's inequality and which works for any disorder distribution with the only condition (stability) that the quenched specific pressure is bounded.

Keywords: Thermodynamic limit; quantum spin glasses; free energy; subadditivity.

1. Introduction, Definitions and Results

In this paper we study the problem of the existence of the thermodynamic limit for a wide class of disordered models defined on finite dimensional lattices. We consider both the classical and quantum case with random two-body or multi-body interaction. The classical case has been studied in various places (see for example [4–8]). In [4] and [7], the quantum case with pair interactions has also been considered. Here we deal only with the quenched pressure. Using only thermodynamic convexity and a mild stability condition, we give a very simple proof of the existence and monotonicity of the quenched specific pressure. A result in the same spirit for classical spin glasses has been obtained in [1] by using an interpolation technique introduced in [2, 3]. The present work extends the results of [1] not only to the quantum case, but also to the classical case with a nonzero mean of the interaction and to the continous spin space.

We shall treat the classical and quantum cases in parallel. In the classical case to each point of the lattice $i \in \mathbb{Z}^d$, we associate a copy of the *spin space* $\mathcal{S}$, which is equipped with an *a priori* probability measure μ. We shall denote this by $\mathcal{S}_i$. In the quantum analogue, we associate to each $i \in \mathbb{Z}^d$ a copy of a finite dimensional Hilbert space $\mathcal{H}$, denoted by $\mathcal{H}_i$ and a set of self-adjoint operators, *spin operators*, on $\mathcal{H}_i$.

Following [9] (see also [10]), we define the interaction in the following way. In the classical case for each finite subset of $\mathbb{Z}^d$, X, we let $\mathcal{S}_X := \times_{i\in X}\mathcal{S}_i$ and $\{\Phi_X^{(j)} \mid j \in n_X\}$ is a finite set of bounded functions from $\mathcal{S}_X$ to $\mathbb{R}$ which are measurable with respect to the product measure $\mu^{|X|}$ on $\mathcal{S}_X$. In the quantum case, each $\Phi_X^{(j)}$ is a self-adjoint element of the algebra generated by the set of operators, the *spin operators* on $\mathcal{H}_X := \otimes_{i\in X}\mathcal{H}_i$. Without loss of generality, we set $\Phi_\emptyset = 0$. In both cases, we take the interaction to be translation invariant in the sense that if τ_a is translation by $a \in \mathbb{Z}^d$, then

$$n_{\tau_a X} = n_X \text{ and } \Phi_{\tau_a X}^{(j)} = \tau_a \Phi_X^{(j)} \text{ for } j \in n_X\,. \tag{1}$$

We now define the random coefficients. For each X, let $\{J_X^{(j)} \mid j \in n_X\}$ be a set of random variables. We assume that the $J_X^{(j)}$'s are independent random variables and that $J_{\tau_a X}^{(j)}$ and $J_X^{(j)}$ have the same distribution for all $a \in \mathbb{Z}^d$. We shall denote the average over the J's by $\mathrm{Av}[\cdot]$.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite set of a regular lattice in d dimensions and denote by $|\Lambda| = N$ its cardinality. We define the *random potential* as

$$U_\Lambda(J,\Phi) := \sum_{X\subset\Lambda}\sum_{j\in n_X} J_X^{(j)}\Phi_X^{(j)}\,. \tag{2}$$

We stress here that the distributions of the $J_X^{(j)}$'s are independent of the volume Λ. This characterizes the short range case, such as the Edwards–Anderson model. In mean field (long range) models, such as the Sherrington–Kirkpatrick model in which the Hamiltonian sums over all the couples (N^2 terms), the variance of $J_X^{(j)}$ has to decrease like N^{-1} in order to have a well defined thermodynamic behavior and in particular a finite energy density. The complete definition of the model we are considering requires that we specify also the interaction on the frontier $\partial\Lambda$, i.e. boundary conditions. However, standard surface over volume arguments imply that if the quenched specific pressure for one boundary condition converges, then it also converges for all other boundary conditions. Therefore, to prove the convergence of the quenched specific pressure, it is sufficient to consider the free boundary condition. Thus in the sequel, we shall assume the free boundary condition and prove that in this case the quenched pressure is monotonically increasing in the volume.

We would like to emphasize the fact that in the classical case, our results are not restricted to the situation when the space $\mathcal{S}$ consists of a finite number of points. Here we also want to cover the case of continuous spins and therefore we shall keep the classical and quantum cases separate. Of course, both cases can be covered simultaneously in a C^* algebra setting but for the sake of simplicity, we shall not take this route.

Example 1 (Classical Edwards–Anderson Model). $\mathcal{S} = \{-1, 1\}$, $\mu(\sigma_i) = \frac{1}{2}\delta(\sigma_i + 1) + \frac{1}{2}\delta(\sigma_i - 1)$. The interaction is only between nearest neighbors: $\Phi_{i,j}(\sigma_i, \sigma_j) = \sigma_i\sigma_j$ for $|i - j| = 1$, $\Phi_X = 0$ otherwise. To ensure that the specific pressure is bounded, it is enough that

$$\mathrm{Av}[|J_{ij}|] < \infty\,. \tag{3}$$

More generally, one may consider a long range interaction with $\Phi_{i,j}(\sigma_i, \sigma_j) = \sigma_i\sigma_j/R(|i - j|)$ with a sufficient condition for boundedness, for example

$$\mathrm{Av}[J_{0i}] = 0 \text{ and } \sum_i \frac{\mathrm{Av}[|J_{0i}|^2]}{(R(|i|))^2} < \infty\,, \tag{4}$$

or a many-body interaction with a suitable decay law. One can also add a (random) external field.

We refer the reader to [1] for more classical examples.

Example 2 (Quantum Edward–Anderson Model). $\mathcal{H} = \mathbb{C}^2$. The spin operators are the set of the Pauli matrices: $\sigma_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

with commutation and anticommutation relations

$$[\,\sigma_i^\alpha, \sigma_i^\beta\,] = 2i\epsilon_{\alpha\beta\gamma}\sigma_i^\gamma\,, \tag{6}$$

$$\{\sigma_i^\alpha, \sigma_i^\beta\} = 2\delta_{\alpha\beta}\,. \tag{7}$$

The interaction is again only between nearest neighbors: $\Phi_{i,j}(\sigma_i, \sigma_j) = \sigma_i \cdot \sigma_j = \sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y + \sigma_i^z\sigma_j^z$ for $|i-j| = 1$, $\Phi_X = 0$ otherwise. A transverse field $\Phi_i(\sigma_i) = \sigma_i^z$ can also be added. One can have an asymmetric version with local interaction

$$J_{i,j}^x\Phi_{i,j}^x(\sigma_i, \sigma_j) + J_{i,j}^y\Phi_{i,j}^y(\sigma_i, \sigma_j) + J_{i,j}^z\Phi_{i,j}^z(\sigma_i, \sigma_j)\,, \tag{8}$$

where $\Phi_{i,j}^x(\sigma_i, \sigma_j) = \sigma_i^x\sigma_j^x$, $\Phi_{i,j}^y(\sigma_i, \sigma_j) = \sigma_i^y\sigma_j^y$ and $\Phi_{i,j}^z(\sigma_i, \sigma_j) = \sigma_i^z\sigma_j^z$. As in Example 1, one may consider a short range interaction with a suitable decay law.

Notation. We shall use the notation Tr to denote both the classical expectation over $\mathcal{S}^N$ with the measure $\mu(d\sigma) = \prod_{i=1}^N \mu(d\sigma_i)$ and the usual trace in quantum mechanics on the Hilbert space $\otimes_{i=1}^N \mathcal{H}$.

Definition 1. We define in the usual way:

(1) the random partition function, $Z_\Lambda(J)$, by

$$Z_\Lambda(J) := \mathrm{Tr}\, e^{U_\Lambda(J,\Phi)}\,; \tag{9}$$

(2) the quenched pressure, P_Λ, by

$$P_\Lambda := \mathrm{Av}[\ln Z_\Lambda(J)]\,; \tag{10}$$

(3) the quenched specific pressure, p_Λ, by

$$p_\Lambda := \frac{P_\Lambda}{N} \,. \tag{11}$$

We are now ready to state our main theorem as follows.

Theorem 1. *If all the* $J_X^{(j)}$*'s with* $|X| > 1$ *have zero mean, then the quenched pressure is superadditive*

$$P_\Lambda \geq \sum_{s=1}^{n} P_{\Lambda_s} \,. \tag{12}$$

Let $\|\Phi_X^{(j)}\|$ denote the supremum norm in the classical case and the operator norm in the quantum case. For the case when the $J_X^{(j)}$'s do not have zero mean, we have the following corollary.

Corollary 1. *Let*

$$\bar{P}_\Lambda = P_\Lambda + \sum_{X\subset\Lambda,\,|X|>1}\sum_{j\in n_X} |\mathrm{Av}[J_X^{(j)}]|\,\|\Phi_X^{(j)}\| \,. \tag{13}$$

Then $\bar{P}_\Lambda$ *is superadditive.*

Theorem 1 combined with the boundedness of the specific pressure is sufficient to ensure the convergence of the specific pressure in the thermodynamic limit (see for example [9, Chap. IV]) in the case when all the $J_X^{(j)}$'s with $|X| > 1$ have zero mean. In the case when the $J_X^{(j)}$'s do not have zero mean, we have to add to Corollary 1 the condition

$$C := \sum_{X\ni 0,\,|X|>1}\sum_{j\in n_X} \frac{|a_X^{(j)}|\|\Phi_X^{(j)}\|}{|X|} < \infty \,. \tag{14}$$

This implies that

$$\lim_{\Lambda\to\infty} \frac{1}{N} \sum_{X\subset\Lambda,\,|X|>1}\sum_{j\in n_X} |a_X^{(j)}|\|\Phi_X^{(j)}\| = C \tag{15}$$

and therefore the convergence of the specific pressure.

To prove the boundedness of the specific pressure, we need the following stability condition (cf. [8]). Let

$$\|U\|_1 := \sum_{X\ni 0}\sum_{j\in n_X} \frac{\mathrm{Av}\big[|J_X^{(j)}|\big]\,\|\Phi_X^{(j)}\|}{|X|} \tag{16}$$

and

$$\|U\|_2 := \left(\sum_{X\ni 0}\sum_{j\in n_X} \frac{\mathrm{Av}[|J_X^{(j)}|^2]\|\Phi_X^{(j)}\|^2}{|X|}\right)^{\frac{1}{2}} \,. \tag{17}$$

Definition 2. We shall say that the random potential $U(J, \Phi)$ is stable if it is of the form

$$U_\Lambda(J, \Phi) = \tilde{U}_\Lambda(\tilde{J}, \tilde{\Phi}) + \hat{U}_\Lambda(\hat{J}, \hat{\Phi}) \,, \tag{18}$$

where all the $\tilde{J}_X^{(j)}$'s and $\hat{J}_X^{(j)}$'s are independent, the $\hat{J}_X^{(j)}$'s have zero mean and $\|\tilde{U}\|_1$ and $\|\hat{U}\|_2$ are finite.

With this definition, we shall prove in the next theorem that the specific pressure is bounded. Note that the stability condition in Definition 2 implies that C as defined in (14) is finite since $C \leq \|U\|_1$.

Theorem 2. *For a stable random potential, the quenched specific pressure is bounded.*

In the next section we prove the theorems.

2. Proof of the Theorems

We start with the following definition.

Definition 3. Consider a partition of Λ into n nonempty disjoint sets Λ_s

$$\Lambda = \bigcup_{s=1}^{n} \Lambda_s \,, \tag{19}$$

$$\Lambda_s \cap \Lambda_{s'} = \emptyset \,. \tag{20}$$

For each partition, the potential generated by all interactions among different subsets is defined as

$$\tilde{U}_\Lambda := U_\Lambda - \sum_{s=1}^{n} U_{\Lambda_s} \,. \tag{21}$$

From (2) it follows that

$$\tilde{U}_\Lambda = \sum_{X \in \mathcal{C}_\Lambda} \sum_{j \in n_X} J_X^{(j)} \Phi_X^{(j)} \,, \tag{22}$$

where $\mathcal{C}_\Lambda$ is the set of all $X \subset \Lambda$ which are not subsets of any Λ_s.

The idea here is to eliminate $\tilde{U}_\Lambda$ from the partition function. We shall use the following three lemmas.

Lemma 1. *Let $X_1, \ldots, X_n$ be independent random variables with zero mean. Let $F\colon \mathbb{R}^n \mapsto \mathbb{R}$ be such that for each $i = 1, \ldots, n$, $x_i \mapsto F(x_1, \ldots, x_n)$ is convex, then*

$$\mathbb{E}[F(X_1, \ldots, X_n)] \geq F(0, \ldots, 0) \,, \tag{23}$$

where $\mathbb{E}$ denotes the expectation with respect to $X_1, \ldots, X_n$.

Proof. This follows by applying Jensen's Inequality to each X_i successively. □

The following two lemmas are related to the thermodynamic convexity of the pressure.

Lemma 2. *Let μ be a probability measure on a space Ω, and let A and $B_1, \ldots, B_n$ be measurable real-valued functions on Ω. Then*

$$\mathbb{E}\left[\log \int_\Omega \exp\left\{A(\sigma) + \sum_{i=1}^n X_i B_i(\sigma)\right\} \mu(d\sigma)\right] \geq \log \int_\Omega \exp[A(\sigma)]\mu(d\sigma)\,. \tag{24}$$

Proof. We just have to check that if

$$F(x_1, \ldots, x_n) = \log \int_\Omega \exp\left\{A(\sigma) + \sum_{i=1}^n x_i B_i(\sigma)\right\} \mu(d\sigma)\,,$$

then $x_i \mapsto F(x_1, \ldots, x_n)$ is convex. Let

$$\langle C\rangle := \frac{\int_\Omega C(\sigma)\exp\{A(\sigma) + \sum_{i=1}^n x_i B_i(\sigma)\}\,\mu(d\sigma)}{\int_\Omega \exp\{A(\sigma) + \sum_{i=1}^n x_i B_i(\sigma)\}\,\mu(d\sigma)}\,. \tag{25}$$

Then, computing the derivatives, we have

$$\frac{\partial F}{\partial x_i} = \langle B_i\rangle \tag{26}$$

and

$$\frac{\partial^2 F}{\partial x_i^2} = \left\langle B_i^2\right\rangle - \langle B_i\rangle^2 = \left\langle\left(B_i - \langle B_i\rangle\right)^2\right\rangle \geq 0\,. \tag{27}$$

□

The next lemma is the quantum analogue of the previous one.

Lemma 3. *Let $\mathcal{H}$ be finite-dimensional Hilbert space, and let A and $B_1, \ldots, B_n$ be self-adjoint operators on $\mathcal{H}$. Then*

$$\mathbb{E}\left[\log \mathrm{Tr}\, \exp\left(A + \sum_{i=1}^n X_i B_i\right)\right] \geq \log \mathrm{Tr}\, \exp A\,. \tag{28}$$

Proof. Again we just have to check that if

$$F(x_1, \ldots, x_n) = \log \mathrm{Tr}\, \exp\left(A + \sum_{i=1}^n x_i B_i\right),$$

then $x_i \mapsto F(x_1, \ldots, x_n)$ is convex. The first derivative gives

$$\frac{\partial F}{\partial x_i} = \langle B_i\rangle\,, \tag{29}$$

where

$$\langle C\rangle := \frac{\mathrm{Tr}\, Ce^{-H}}{\mathrm{Tr}\, e^{-H}} \tag{30}$$

with

$$-H = A + \sum_{i=1}^{n} x_i B_i$$

while, for the second derivative, we have

$$\frac{\partial^2 F}{\partial x_i^2} = (B_i, B_i) - \langle B_i \rangle^2 \,, \tag{31}$$

where $(\cdot,\cdot)$ denotes the Du Hamel inner product (see for example [10])

$$(C, D) := \frac{\mathrm{Tr}\int_0^1 ds\ e^{-sH} C^* e^{(1-s)H} D}{\mathrm{Tr}\, e^{-H}}. \tag{32}$$

By using the fact that $(C, 1) = \overline{\langle C \rangle}$ and $(1, D) = \langle D \rangle$, we see that

$$\frac{\partial^2 F}{\partial x_i^2} = (B_i - \langle B_i \rangle, B_i - \langle B_i \rangle) \geq 0 \,. \tag{33}$$

□

Proof of Theorem 1. Let us assume first that all the $J_X^{(j)}$'s with $|X| > 1$ have zero mean.

$$\begin{aligned} P_\Lambda &= \mathrm{Av}[\ln \mathrm{Tr}\, \exp U_\Lambda] \\ &= \mathrm{Av}\left[\ln \mathrm{Tr}\, \exp\left(\sum_{s=1}^{n} U_{\Lambda_s} + \sum_{X \in \mathcal{C}_\Lambda} \sum_{j \in n_X} J_X^{(j)} \Phi_X^{(j)}\right)\right]. \end{aligned} \tag{34}$$

Note that $\mathcal{C}_\Lambda$ does not contain any X with $|X| = 1$. Applying Lemma 2 (resp. Lemma 3) for the classical (resp. quantum) case with $A = \sum_{s=1}^{n} U_{\Lambda_s}$, $B_i = \Phi_X^{(j)}$ and $n = \sum_{X \in \mathcal{C}_\Lambda} n_X$, we get

$$P_\Lambda \geq \mathrm{Av}\left[\ln \mathrm{Tr}\, \exp\left(\sum_{s=1}^{n} U_{\Lambda_s}\right)\right] = \sum_{s=1}^{n} \mathrm{Av}[\ln \mathrm{Tr}\, \exp U_{\Lambda_s}] = \sum_{s=1}^{n} P_{\Lambda_s} \,. \tag{35}$$

□

Proof of Corollary 1. Here we relax the condition that all the J's have zero mean. Let $a_X^{(j)} := \mathrm{Av}[J_X^{(j)}]$ and $\bar{J}_X^{(j)} := J_X^{(j)} - a_X^{(j)}$ for $|X| > 1$, so that $\bar{J}_X^{(j)}$ has zero mean and $\bar{J}_X^{(j)} := J_X^{(j)}$ if $|X| = 1$. Let

$$U_\Lambda^{(1)}(J, \Phi) := \sum_{X \subset \Lambda} \sum_{j \in n_X} \bar{J}_X^{(j)} \Phi_X^{(j)} \,, \tag{36}$$

$$U_\Lambda^{(2)}(J, \Phi) := \sum_{X \subset \Lambda,\, |X| > 1} \sum_{j \in n_X} \left(a_X^{(j)} \Phi_X^{(j)} + |a_X^{(j)}| \|\Phi_X^{(j)}\|\right) \tag{37}$$

and

$$\bar{U}_\Lambda(J, \Phi) := U_\Lambda^{(1)}(J, \Phi) + U_\Lambda^{(2)}(J, \Phi) \,. \tag{38}$$

Then

$$\bar{U}_\Lambda(J,\Phi) = U_\Lambda(J,\Phi) + \sum_{X\subset\Lambda,\,|X|>1}\sum_{j\in n_X} |a_X^{(j)}|\|\Phi_X^{(j)}\|\,. \tag{39}$$

Thus $\bar{P}_\Lambda$ is the pressure corresponding to $\bar{U}_\Lambda(J,\Phi)$. One can then see that $\bar{P}_\Lambda$ is superadditive by treating the terms in $U_\Lambda^{(1)}(J,\Phi)$ as before, since each $\bar{J}_X^{(j)}$ has zero mean, except possibly if $|X|=1$, and by using the fact that all the terms in $U_\Lambda^{(2)}(J,\Phi)$ are positive (cf. [10]). In the quantum case, we need the inequality

$$\operatorname{Tr} e^{(A+B)} \geq \operatorname{Tr} e^{A} \tag{40}$$

if B is a positive operator. □

Proof of Theorem 2. The proof in the classical case is given in [8]. Here we modify that proof to cover the quantum case. From the Bogoliubov inequality

$$\frac{\operatorname{Tr}(A-B)e^B}{\operatorname{Tr} e^B} \leq \ln\operatorname{Tr} e^A - \ln\operatorname{Tr} e^B \leq \frac{\operatorname{Tr}(A-B)e^A}{\operatorname{Tr} e^A} \tag{41}$$

with $A = U_\Lambda(J,\Phi)$ and $B=0$ we get

$$\begin{aligned}\log Z_\Lambda(J) - N\log\dim\mathcal{H} &\leq \frac{\operatorname{Tr} U_\Lambda(J,\Phi)e^{U_\Lambda(J,\Phi)}}{\operatorname{Tr} e^{U_\Lambda(J,\Phi)}}\\ &= \frac{\operatorname{Tr}\tilde{U}_\Lambda(\tilde{J},\tilde{\Phi})e^{U_\Lambda(J,\Phi)}}{\operatorname{Tr} e^{U_\Lambda(J,\Phi)}} + \frac{\operatorname{Tr}\hat{U}_\Lambda(\hat{J},\hat{\Phi})e^{U_\Lambda(J,\Phi)}}{\operatorname{Tr} e^{U_\Lambda(J,\Phi)}}\\ &\leq \|\tilde{U}_\Lambda(\tilde{J},\tilde{\Phi})\| + \frac{\operatorname{Tr}\hat{U}_\Lambda(\hat{J},\hat{\Phi})e^{U_\Lambda(J,\Phi)}}{\operatorname{Tr} e^{U_\Lambda(J,\Phi)}}\,.\end{aligned} \tag{42}$$

Now

$$\operatorname{Av}[\|\tilde{U}_\Lambda(\tilde{J},\tilde{\Phi})\|] \leq N\|\tilde{U}(\tilde{J},\tilde{\Phi})\|_1\,. \tag{43}$$

For the other term, we use the identity for A and B self-adjoint

$$\frac{\operatorname{Tr} Ae^{A+B}}{\operatorname{Tr} e^{A+B}} - \frac{\operatorname{Tr} Ae^{B}}{\operatorname{Tr} e^{B}} = \int_0^1 dt(A-\langle A\rangle_t,\, A-\langle A\rangle_t)_t\,, \tag{44}$$

where $\langle\cdot\rangle_t$ and $(\cdot,\cdot)_t$ denote the mean and the Du Hamel inner product, respectively with respect to $H=-(tA+B)$. The Du Hamel inner product satisfies

$$(C,C) \leq \frac{1}{2}\langle C^*C + CC^*\rangle^{\frac{1}{2}} \leq \|C\|^2\,. \tag{45}$$

Therefore

$$\frac{\operatorname{Tr} Ae^{A+B}}{\operatorname{Tr} e^{A+B}} - \frac{\operatorname{Tr} Ae^{B}}{\operatorname{Tr} e^{B}} \leq 4\|A\|^2\,. \tag{46}$$

With $A = \hat{J}^j_X \hat{\Phi}^j_X$ and $B = U_\Lambda(J,\Phi) - \hat{J}^j_X \hat{\Phi}^j_X$ we get

$$\frac{\operatorname{Tr} \hat{U}_\Lambda(\hat{J},\hat{\Phi}) e^{U_\Lambda(J,\Phi)}}{\operatorname{Tr} e^{U_\Lambda(J,\Phi)}} = \sum_{X \subset \Lambda} \sum_{j \in \hat{n}_X} \frac{\operatorname{Tr} \hat{J}^j_X \hat{\Phi}^j_X e^{U_\Lambda(J,\Phi)}}{\operatorname{Tr} e^{U_\Lambda(J,\Phi)}}$$

$$\leq \sum_{X \subset \Lambda} \sum_{j \in \hat{n}_X} \operatorname{Tr} \hat{J}^j_X \hat{\Phi}^j_X \frac{e^{U_\Lambda(J,\Phi) - \hat{J}^j_X \hat{\Phi}^j_X}}{\operatorname{Tr} e^{U_\Lambda(J,\Phi) - \hat{J}^j_X \hat{\Phi}^j_X}}$$

$$+ 4 \sum_{X \subset \Lambda} \sum_{j \in \hat{n}_X} |\hat{J}^j_X|^2 \|\hat{\Phi}^j_X\|^2 . \tag{47}$$

Thus since $U_\Lambda(J,\Phi) - \hat{J}^j_X \hat{\Phi}^j_X$ is independent of $\hat{J}^j_X$ and $\operatorname{Av}\big[\hat{J}^j_X\big] = 0$,

$$\operatorname{Av}\left[\frac{\operatorname{Tr} \hat{U}_\Lambda(\hat{J},\hat{\Phi}) e^{U_\Lambda(J,\Phi)}}{\operatorname{Tr} e^{U_\Lambda(J,\Phi)}}\right] \leq 4 \sum_{X \subset \Lambda} \sum_{j \in \hat{n}_X} \operatorname{Av}\big[|\hat{J}^j_X|^2\big] \|\hat{\Phi}^j_X\|^2 \leq 4N \|\hat{U}(\hat{J},\hat{\Phi})\|_2^2 . \tag{48}$$

Therefore

$$P_\Lambda \leq N(\log \dim \mathcal{H} + \|\tilde{U}(\tilde{J},\tilde{\Phi})\|_1 + 4\|\hat{U}(\hat{J},\hat{\Phi})\|_2^2) . \tag{49}$$

$\square$

Acknowledgments

The authors wish to thank B. Nachtergaele and A. van Enter for very constructive suggestions. One of the authors (J. Pulé) wishes to thank the Department of Mathematics of the University of Bologna, Italy, for their kind hospitality, and University College Dublin for the award of a President's Fellowship. P. Contucci and C. Giardinà wish to thank S. Graffi for some very useful discussions.

References

[1] P. Contucci and S. Graffi, *J. Statist. Phys.* **115**(1/2) (2004) 581–589.
[2] F. Guerra and F. Toninelli, *Commun. Math. Phys.* **230** (2002) 71–79.
[3] P. Contucci, M. Degli Esposti, C. Giardinà and S. Graffi, *Commun. Math. Phys.* **236** (2003) 55–63.
[4] P. A. Vuillermot, *J. Phys. A*: *Math. Gen.* **10** (1977) 1319–1333.
[5] L. A. Pastur and A. L. Figotin, *Theor. Math. Phys.* **35** (1978) 193–202.
[6] K. M. Khanin and Ya. G. Sinai, *J. Statist. Phys.* **20** (1979) 573–584.
[7] A. C. D. van Enter and J. L. van Hemmen, *J. Statist. Phys.* **32** (1983) 141–152.
[8] B. Zegarlinski, *Commun. Math. Phys.* **139** (1991) 305–339.
[9] D. Ruelle, *Statistical Mechanics, Rigorous Results* (W. A. Benjamin, New York, 1969).
[10] B. Simon, *The Statistical Mechanics of Lattice Gases* (Princeton, University Press, 1992).